MATH136 - Discussion Supplements for Spring 22

Some of the contents are motivated from [1] and the lecture notes for Math 136 (for Spring 22) by Marcus Roper. [1]

Contents

1 Discussion 1.

136 Pre-requisite Materials

Partial Derivatives.

Definition 1. Consider a function of two variables $f(x, y)$, $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ corresponds to the domain of the function. We say that the partial derivatives f_x or $\frac{\partial f}{\partial x}$ of f with respect to x at a point $(a, b) \in D$ exists (and is defined as the value of the following limit) if the following limit exists:

$$f_x(a,b) = \frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}. \tag{1}$$

Similarly, we can define f_y or $\frac{\partial f}{\partial y}$ as follows:

$$f_y(a,b) = \frac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}. \tag{2}$$

Computationally, as observed from the limits above, $\frac{\partial f}{\partial x}$ is the same as differentiating f with respect to x, **while keeping y constant**. Such a concept generalizes to a function with $n \in \mathbb{N}$ variables, in which partially differentiating with respect to one of the variables can be done computationally by keeping the other variables constant. The following example illustrates some of these computation:

Example 2.

(a) Let $f(x, y) = x^2 y$. Compute $\frac{\partial f}{\partial y}$.

(b) Let $f(x, y, z) = x^2 y + e^z \sin(y)$. Compute $\frac{\partial f}{\partial z}(1, 1, 1)$.

Suggested Solution:

(a) $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 y) = x^2$ (by treating x as a constant when we are differentiating with respect to y).

(b) $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2 y + e^z \sin(y)) = e^z \sin(y)$ (by treating x and y as constants when we are differentiating with respect to z). Thus, we can compute $\frac{\partial f}{\partial z}$ by plugging in $(x, y, z) = (1, 1, 1)$ into our expression above to obtain:

$$\frac{\partial f}{\partial z}(1,1,1) = e^1 \sin(1).$$

Example 3. (A partial derivative riddle.) A student propose the following computation for computing partial derivatives of the Cartesian coordinates (x, y) in $\mathbb{R}^2$ with respect to the polar coordinates variables (r, θ) as follows:

136 Student in Another Dimension (since this student is unlikely to make this mistake after today!):
Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we can compute

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos(\theta)) = \cos(\theta) = \frac{x}{r}. \tag{3}$$

However, we can also show that $r^2 = x^2 + y^2$, so taking $\frac{\partial}{\partial r}$ on both sides of the equation, we obtain

$$2r = \frac{\partial}{\partial r}(x^2 + y^2) = 2x\frac{\partial x}{\partial r}. \tag{4}$$

From (3) and (4), we obtain

$$\frac{\partial x}{\partial r} = \frac{x}{r} = \frac{r}{x}, \tag{5}$$

which seems to be contradictory. What went wrong here?

Suggested Solution: We have to be careful as to **"what is kept constant"**. In (3), when we take $\frac{\partial}{\partial r}$, the variable kept constant is θ (since we view x as a function of two variables, θ and r). In (4), when we take $\frac{\partial}{\partial x}$, the variable kept constant is y. Thus, if we note down the variable that is kept constant in our computation (and record it as a subscript - see the equations that follow), we have a slightly more rigorous way of writing (3) and (4) as follows:

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \frac{x}{r}, \tag{6}$$

while

$$\left(\frac{\partial x}{\partial r}\right)_y = \frac{r}{x}. \tag{7}$$

Then, it is clear that our equality in (5) was not exactly an "equality" per se, that is,

$$\frac{x}{r} = \left(\frac{\partial x}{\partial r}\right)_\theta \text{(not necessarily } = \text{)} \left(\frac{\partial x}{\partial r}\right)_y = \frac{r}{x}. \tag{8}$$

Explicitly, in (4), if we have kept the same variable θ constant, it should have been

$$2r = \left(\frac{\partial}{\partial r}\right)_\theta (x^2 + y^2) = 2x\left(\frac{\partial x}{\partial r}\right)_\theta + 2y\left(\frac{\partial y}{\partial r}\right)_\theta. \tag{9}$$

Multivariable Calculus Operators.

Definition 4. Let $f : D \to \mathbb{R}$ and $\mathbf{F} : D \to \mathbb{R}^n$ (ie a vector-valued function), where $D \subset \mathbb{R}^n$ is the domain of the functions. Note that we can write $f(x_1, ..., x_n)$ and $\mathbf{F}(x_1, ..., x_n) = \begin{pmatrix} F_1(x_1, ..., x_n) \\ \vdots \\ F_n(x_1, ..., x_n) \end{pmatrix}$, where F_i are the components of the vector(-valued function).

- The **gradient**[a] of f is defined as
$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right\rangle.$$
- Given a **unit** normal vector $\mathbf{n}$, we define the **directional derivative** of f in the direction of $\mathbf{n}$ as
$$\frac{\partial f}{\partial n} = \mathbf{n} \cdot \nabla f.$$
- The **divergence** of a vector-valued function $\mathbf{F}$ is given by
$$\nabla \cdot \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}.$$
- The **Laplacian**[b] of a scalar function f is given by
$$\nabla^2 f = \Delta f = \nabla \cdot (\nabla f) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$
Note that taking the second derivative $\frac{\partial^2}{\partial x^2}$ means taking the partial derivative $\frac{\partial}{\partial x}$ twice.
- A scalar function f is **harmonic** if $\nabla^2 f = 0$.

[a] Sometimes, we write a vector as $\langle ... \rangle$ as a row vector, or $\begin{pmatrix} \vdots \end{pmatrix}$ as a column vector. This is completely dependent on the the reader.
[b] Some books write it as ∇^2, others write it as Δ.

Example 5.

(a) Let $f(x, y) = x^2 y$. Compute ∇f.

(b) Let $f(x, y) = x^2 y$, and $\mathbf{n} = \langle 1, 0 \rangle$. Compute $\frac{\partial f}{\partial n}$.

(c) Let $f(x, y, z) = x^2 y + e^z \sin(y)$. Compute $\nabla^2 f$.

Suggested Solution:

(a) $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle 2xy, x^2 \rangle$.

(b) $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle 2xy, x^2 \rangle$. Hence, $\frac{\partial f}{\partial n} = \mathbf{n} \cdot \nabla f = \langle 1, 0 \rangle \cdot \langle 2xy, x^2 \rangle = 2xy$.
Note that this makes intuitive sense since the directional derivative along $\langle 1, 0 \rangle$ basically means the partial derivative of f along the $x-$direction.

(c) $\frac{\partial^2 f}{\partial x^2} = 2y$, $\frac{\partial^2 f}{\partial y^2} = x^2 + e^z \cos(y)$ and $\frac{\partial^2 f}{\partial z^2} = e^z \sin(y)$.
This implies that $\nabla^2 f = x^2 + 2y + e^z(\sin(y) + \cos(y))$.

Example 6. (Laplacian in $\mathbb{R}^n$.) Compute $\nabla^2(r^2)$, where $r = \sqrt{x_1^2 + ... + x_n^2}$.

Suggested Solution: Note that for a given $i \in \{1, ..., n\}$, we have

$$\frac{\partial r^2}{\partial x_i} = \frac{\partial}{\partial x_i}(x_1^2 + ... + x_n^2) = 2x_i.$$

Thus,

$$\frac{\partial^2 r^2}{\partial x_i^2} = 2\frac{\partial}{\partial x_i}x_i = 2.$$

Summing them up, we have

$$\nabla^2(r^2) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i}r^2 = \sum_{i=1}^{n} 2 = 2n.$$

Chain Rule.
Let $f : D \to \mathbb{R}$ with $D \subset \mathbb{R}^2$, written as $f(x, y)$. Suppose that x and y are functions of two variables, ie $x : D_x \to \mathbb{R}^2$ and $y : D_y \to \mathbb{R}^2$ for some $D_x, D_y \subset \mathbb{R}^2$, written as $x(u, v)$ and $y(u, v)$. Then, we have

$$\frac{\partial}{\partial u}f(x(u, v), y(u, v)) = \frac{\partial f(x, y)}{\partial x}\frac{\partial x(u, v)}{\partial u} + \frac{\partial f(x, y)}{\partial y}\frac{\partial y(u, v)}{\partial u} \tag{10}$$

One way to understand the above is this:

- You want to reach u via x and y.
- Since both x and y are connected to u, you can do this in two ways.
- One pathway: $f \to x \to u$. The other: $f \to y \to u$.
- Do a simple chain rule along each pathway and add them up.

Why does it work? Looking at our chain rule above, see that when we take $\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}$, this is nothing but chain rule if it was $f(x(u))$ and that we were taking $\frac{\mathrm{d}}{\mathrm{d}u}$. However, since the way we have written $f(x(u))$ is to treat y and v as constants, these are "implicit" at each step of the computation.

To be notationally complete, using our notation in Example 3, we have

$$\left(\frac{\partial f(x(u, v), y(u, v))}{\partial u}\right)_v = \left(\frac{\partial f(x, y)}{\partial x}\right)_y \left(\frac{\partial x(u, v)}{\partial u}\right)_v + \left(\frac{\partial f(x, y)}{\partial y}\right)_x \left(\frac{\partial y(u, v)}{\partial u}\right)_v \tag{11}$$

A remark here would be that most mathematical literature do not include the explicit variables kept constant as it is “obvious” to them. However, for applications arising from physical model (especially so in thermodynamics), it is extremely important to keep track of the variables kept constant!

Example 7.

(a) Let $f(x,t) = x^2 - t^2$ and $x(u,v) = e^{u+v}$ and $t(u,v) = e^{u-v}$. Compute $\frac{\partial f}{\partial v}$ and express it in terms of u and v.

(b) Let $f(x,y) = x^2 - y^2$ and $x(u,v) = e^{u+v}$ and $y(u) = u$. Compute $\frac{\partial f}{\partial v}$ and express it in terms of u and v.

Suggested Solutions:

(a) Since we have $f \to x \to v$ and $f \to t \to v$, then

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial v} = 2xe^{u+v} + 2te^{u-v} = 2e^{2(u+v)} + 2e^{2(u-v)}.$$

(b) We can either apply (11) directly, or derive our new chain rule with the new set of dependence. For this example, I'll illustrate the second method. Since the only pathway to v is by $f \to x \to v$ (as y is independent of v), then, we have

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} = 2xe^{u+v} = 2e^{2(u+v)}. \tag{12}$$

Example 8. (A chain rule in $\mathbb{R}^n$.) Suppose that $f(x_1, ..., x_n)$ and for each $i \in \{1, ..., n-1\}$, $x_i(u)$ (ie is a function of one variable, say u), and $x_n(u,v)$ (x_n is a function of two variables, u and v). Compute

$$\frac{\partial f}{\partial v}.$$

Suggested Solution: Note that the only pathway from f to v is via $f \to x_n \to v$ since x_i for $i \in \{1, ..., n-1\}$ does not depend on v. This implies that

$$\frac{\partial f(x_1(u), x_2(u), ..., x_n(u,v))}{\partial v} = \frac{\partial f(x_1, ..., x_n)}{\partial x_n}\frac{\partial x_n(u,v)}{\partial v}. \tag{13}$$

Integrals in Polar Coordinates. Recall in 32B that we can evaluate a multivariate integral in Cartesian coordinates by converting it into one in polar coordinates (or say cylindrical/spherical coordinates in 3D) via a change of variable. Explicitly, let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then, we have

$$\int\int_D f(x,y)\mathrm{d}x\mathrm{d}y = \int\int_{D'} f(x,y)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\mathrm{d}r\mathrm{d}\theta \tag{14}$$

where D' refers to the corresponding domain of integration the function f in polar coordinates, and $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|$ refers to the Jacobian for such a transformation. This can be computed as a determinant of a 2×2 matrix as follows:

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r. \tag{15}$$

Example 9. Compute $\int\int_D 1\mathrm{d}x\mathrm{d}y$ with $D = \{(x,y) : x^2 + y^2 \leq 4\}$.

Suggested Solution: Convert the double integral in polar coordinates. Note that the corresponding domain $D' = \{(r,\theta) : 0 \leq r \leq 2, 0 \leq \theta < 2\pi\}$. Such a domain can be obtained by substituting $x = r\cos\theta$ and $y = r\sin\theta$ into D to obtain $r^2 \leq 4$ which implies $0 \leq r \leq 2$. Furthermore, this does not restrict the values that θ can take. Thus, $0 \leq \theta < 2\pi$. Hence, we have

$$\int\int_D 1\mathrm{d}x\mathrm{d}y = \int\int_D' r\mathrm{d}r\mathrm{d}\theta = \int_0^2\int_0^{2\pi} r\mathrm{d}r\mathrm{d}\theta = \left(\int_0^2 r\mathrm{d}r\right)\left(\int_0^{2\pi}\mathrm{d}\theta\right) = 2 \times 2\pi = 4\pi = \pi(2)^2. \tag{16}$$

This is consistent with our intuition that the area of a circle of radius 2 is given by $\pi(2)^2$.

Divergence Theorem.

Theorem 10. (Divergence Theorem.) If D is a bounded, open set with a piecewise differentiable boundary ∂D, and $\mathbf{f}$ is C^1 (ie continuously differentiable) vector field defined on D, then

$$\int_D \nabla \cdot \mathbf{f} \mathrm{d}V = \int_{\partial D} \mathbf{n} \cdot \mathbf{f} \mathrm{d}S \tag{17}$$

where $\mathbf{n}$ is the unit normal vector that is defined everywhere to point outside of D.

Divergence Theorem has many applications, that includes:

- Evaluating the surface integral on the right of (17).
- Effectively acts as the fundamental theorem of calculus in $\mathbb{R}^n$, which is used to establish integration by parts in $\mathbb{R}^n$ that will be useful for the proof-based part of this class.

Example 11. Let $\mathbf{f}(x, y) = \langle x + y^5, y + x^5 \rangle$ and $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Compute

$$\int_C \mathbf{n} \cdot \mathbf{f} \mathrm{d}S.$$

Suggested Solution: Note that $C = \partial D$, with $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ (the curve serves as the boundary of an open disk of radius 1). By Divergence Theorem, we have

$$\int_C \mathbf{n} \cdot \mathbf{f} \mathrm{d}S = \int_D \nabla \cdot \mathbf{f} \mathrm{d}V.$$

Here, $\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 1 + 1 = 2$. This implies that

$$\int_C \mathbf{n} \cdot \mathbf{f} \mathrm{d}S = \int_D \nabla \cdot \mathbf{f} \mathrm{d}V = 2 \times \int_D \mathrm{d}V = 2\,\mathrm{vol}(D) = 2\pi. \tag{18}$$

Here, volume refers to the 2−dimensional volume, ie area. Furthermore, $\mathrm{vol}(D)$ is the area of a circle with radius 1, and thus $= \pi$.

Example 12. (Proof-based application of Divergence Theorem.) Suppose that $\mathbf{F}$ is conservative (ie there exists a function f such that $\mathbf{F} = \nabla f$) and that the potential function f is harmonic. Let D be a bounded, open set with a piecewise differentiable boundary ∂D. Prove that

$$\int_{\partial D} \mathbf{n} \cdot \mathbf{F} \mathrm{d}S = 0. \tag{19}$$

Suggested Solution: By Divergence Theorem, we have

$$\begin{aligned}
\int_{\partial D} \mathbf{n} \cdot \mathbf{F} \mathrm{d}S &= \int_D \nabla \cdot \mathbf{F} \mathrm{d}V \\
&= \int_D \nabla \cdot (\nabla f) \mathrm{d}V \text{ since } \mathbf{F} \text{ is conservative} \\
&= \int_D \nabla^2 f \, \mathrm{d}V \\
&= \int_D 0 \, \mathrm{d}V \text{ since } f \text{ is harmonic} \\
&= 0.
\end{aligned} \tag{20}$$

2 Discussion 2.

All functions have sufficient smoothness as required, unless stated otherwise.

Recap: Solving ODEs

(i) Separable ODEs, ie. those in the form of

$$\frac{\mathrm{d}}{\mathrm{d}x}y(x) = f(y)g(x).$$

Solution: By separation of variables, we have[2]

$$\int \frac{\mathrm{d}y}{f(y)} = \int g(x)\mathrm{d}x.$$

(ii) General First Order ODEs, ie. those in the form of

$$\frac{\mathrm{d}}{\mathrm{d}x}y(x) = f(x)y(x) + g(x)$$

for some continuous functions $f(x)$ and $g(x)$. This is done via the use of an integrating factor. First, we bring the term with $y(x)$ to the left-hand side to obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}y(x) - f(x)y(x) = g(x).$$

Next, we note that $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\int^x f(s)\mathrm{d}s}\right) = -f(x)e^{-\int^x f(s)\mathrm{d}s}$. Thus, we first multiply both sides of the equation by the *integrating factor* $e^{-\int^x f(s)\mathrm{d}s}$ and observe that the left hand side now factors as an exact differential by consider the two terms as the output of a product rule as follows.

$$\begin{aligned} e^{-\int^x f(s)\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}x}y(x) - e^{-\int^x f(s)\mathrm{d}s}f(x)y(x) &= e^{-\int^x f(s)\mathrm{d}s}g(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\int^x f(s)\mathrm{d}s}y(x)\right) &= e^{-\int^x f(s)\mathrm{d}s}g(x). \end{aligned}$$

Then, by integrating with respect to x, we have

$$\begin{aligned} e^{-\int^x f(s)\mathrm{d}s}y(x) &= \int^x e^{-\int^t f(s)\mathrm{d}s}g(t)\mathrm{d}t + C \\ y(x) &= e^{\int^x f(s)\mathrm{d}s}\left(\int^x e^{-\int^t f(s)\mathrm{d}s}g(t)\mathrm{d}t + C\right). \end{aligned}$$

[2]Notable Point (but not that important for this class): What if $f(y) = 0$ for some y? To argue this in a rigorous sense, the mode of argument is always case-dependent. For instance, if $\frac{\mathrm{d}}{\mathrm{d}x}y(x) = y(x)$, instead of "dividing" by $y(x)$, one could write it as $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x}y(x)\right) = 0$ instead and solve this directly. This completely avoids the "division by zero" issue that we have.

Example 13. (Some ODEs.) Let $x : \mathbb{R} \to \mathbb{R}$ (ie be a function of one variable, say t).

(i) Solve $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{x}$ for $t > 1$, with $x(1) = 1$.

(ii) Solve $\frac{\mathrm{d}x}{\mathrm{d}t} = x + 1$.

Suggested Solutions:

(i) By separation of variables, we bring the x to the other side to obtain

$$\begin{aligned} \int x \mathrm{d}x &= \int 1 \mathrm{d}t \\ \frac{x^2}{2} &= t + C. \end{aligned} \tag{21}$$

Plugging in $x = 1$ at $t = 1$ to obtain $C = -\frac{1}{2}$. Thus, we have

$$x(t) = \sqrt{2t - 1}, t > 1. \tag{22}$$

Here, we pick the positive root to match the initial condition $x(1) = 1$.

(ii) By the method of integrating factor, we first bring x to the other side to obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} - x = 1. \tag{23}$$

Integrating factor (using the formula above) $e^{\int -1 \mathrm{d}t} = e^{-t}$. This implies that

$$\begin{aligned} e^{-t} \frac{\mathrm{d}}{\mathrm{d}t} x - x e^{-t} &= e^{-t} \\ \frac{\mathrm{d}}{\mathrm{d}t}(e^{-t} x) &= e^{-t}. \end{aligned} \tag{24}$$

Integrate with respect to t on both sides to obtain

$$e^{-t} x = -e^{-t} + C \tag{25}$$

or if we bring e^{-t} to the other side, we get

$$x = Ce^t - 1. \tag{26}$$

Here, C is an arbitrary constant.

Example 14. (Notations and Total Derivatives.) Suppose we have a function $f(x,t)$ such that x is also a function of t (ie $x = x(t)$).

(i) Compute $\frac{\mathrm{d}}{\mathrm{d}t} f(x(t), t)$.

(ii) Which of the terms below represent the same function?[a]

$$-\frac{\partial}{\partial t} f(x(t)-t),\ f'(x(t)-t),\ \frac{\partial}{\partial x} f(x(t)-t), \text{ and } \frac{\mathrm{d}}{\mathrm{d}t} f(x(t)-t).$$

Suggested Solutions:

(i) Note that it should be understood from the question that t here is an independent variable, and x, though looks like an independent variable, is actually dependent on t! Note here that the right notation should be $\frac{\mathrm{d}}{\mathrm{d}t} f(x(t), t)$ to emphasize a "total" derivative (we call this the advective derivative in lecture, motivated from Fluid Dynamics). The main/usual way to understand this would be that is a chain rule in disguise, in which since t is the only variable, then we have to write $\frac{\mathrm{d}}{\mathrm{d}t}$ (ie nothing else is really kept constant.) By the chain rule (by thinking of t as a function of t, in which the function is the identity function - $t(t) = t$), we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}. \tag{27}$$

Since $\frac{\partial t}{\partial t} = 1$ and $\frac{\partial x}{\partial t}$ is essentially $\frac{\mathrm{d}x}{\mathrm{d}t}$ (ie nothing else is actually kept constant here, we only have one variable - that is t!), we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}. \tag{28}$$

The equation and the concept above is going to be **extremely** useful for solving first order PDE by method of characteristics for section today and Homework 2!

(ii) First, one should understand what is meant by $f'(x(t)-t)$. This means that we are differentiating the function f with respect to its argument η. Here, $\eta = x - t$. Mathematically, we mean

$$f'(x(t)-t) = \frac{\mathrm{d}f}{\mathrm{d}\eta}.$$

Now, for the first term, we can first write $\eta(x,t) = x - t$. See that $\frac{\partial \eta}{\partial t} = -1$. By the chain rule, we have [b]

$$-\frac{\partial}{\partial t} f(\eta(x,t)) = -\frac{\mathrm{d}f}{\mathrm{d}\eta}\frac{\partial \eta}{\partial t} = \frac{\mathrm{d}f}{\mathrm{d}\eta}. \tag{29}$$

For the third term, we set $\eta(x,t) = x - t$ and note that $\frac{\partial \eta}{\partial x} = 1$. Then, we apply chain rule (similar to the first term) to obtain

$$\frac{\partial}{\partial x} f(\eta(x,t)) = \frac{\mathrm{d}f}{\mathrm{d}\eta}\frac{\partial \eta}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}\eta}. \tag{30}$$

Last but not least, the last term requires the concept of a total derivative. Since $x(t)$, then we can view $\eta(t) = \eta(x(t), t)) = x(t) - t$ (η as a function of t). This implies that $\frac{\partial \eta}{\partial t} = -1$ and $\frac{\partial x}{\partial x} = 1$. We can first apply chain rule to $f(\eta(t))$, and see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(x(t), t) = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

(see (i)). This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\eta(t)) = \frac{\mathrm{d}f}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}\eta}\left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}f}{\mathrm{d}\eta}\left(-1 + 1\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}f}{\mathrm{d}\eta}\left(\frac{\mathrm{d}x}{\mathrm{d}t} - 1\right). \tag{31}$$

An informal way to think about this is to do chain rule directly:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t)-t) = f'(x(t)-t)[x(t)-t]' = f'(x(t)-t)(x'(t)-1). \tag{32}$$

Thus, it is clear that the first three terms mean the same thing, which is not necessarily the same as the last term.

[a]Following our convention in Discussion Supplement 1, it is understood that $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t}\right)_x$ (keeping x constant) and vice versa.

[b]Following how chain rule was covered in the previous discussion, we view $f \to \eta \to x$ or $f \to \eta \to t$ as the only two possible pathways. Note that we stop only at x since it is kept constant (recall the implicit variable kept constant when we say $\frac{\partial}{\partial t}$). Since we would like to take $\frac{\partial}{\partial t}$, then $f \to \eta \to t$ is the only valid pathway!

To get yourself used to the notation, here is an example.

Example 15. Let $x(t) = t^3$ and $f(y) = y^2$. Compute the following:

(i) $-\frac{\partial}{\partial t} f(x(t) - t)$,

(ii) $f'(x(t) - t)$,

(iii) $\frac{\partial}{\partial x} f(x(t) - t)$, and

(iv) $\frac{\mathrm{d}}{\mathrm{d}t} f(x(t) - t)$ by substituting $x(t)$ and computing $\frac{\mathrm{d}}{\mathrm{d}t}$ directly.

For each case, verify the computations directly and by using the chain rule if possible.

Suggested Solution: Note that for partial derivatives, you are **not allowed** to substitute $x(t) = t^3$ for say $\frac{\partial}{\partial t}$ since we should view x as a "constant" first above the fact that it is a function of t (since $\frac{\partial}{\partial x}$ really means $\left(\frac{\partial}{\partial t}\right)_x$, keeping x constant).
Note that $f(x - t) = (x - t)^2$.

(i) Directly: $-\frac{\partial}{\partial t} f(x - t) = -\frac{\partial}{\partial t}(x - t)^2 = 2(x - t)$.
Chain Rule: From the above example, this is equivalent to $\frac{\mathrm{d}f}{\mathrm{d}\eta} = f'(\eta)$. Since $f(y) = y^2$ implies that $f'(y) = 2y$, then $f'(\eta) = f'(x - t) = 2(x - t)$.

(ii) See the above computation under "Chain Rule". $f'(x - t) = 2(x - t)$.

(iii) Directly: $\frac{\partial}{\partial x} f(x - t) = \frac{\partial}{\partial x}(x - t)^2 = 2(x - t)$.
Chain Rule: From the above example, this is equivalent to $\frac{\mathrm{d}f}{\mathrm{d}\eta} = f'(\eta) = f'(x - t) = 2(x - t)$.

(iv) If we say $\frac{\mathrm{d}}{\mathrm{d}t}$, we mean that we are viewing f as a function of only one variable, t.
Directly: $f(x(t) - t) = (x(t) - t)^2 = (t^3 - t)^2 = t^6 - 2t^4 + t^2$.
Thus, $\frac{\mathrm{d}}{\mathrm{d}t} f(x(t) - t) = \frac{\mathrm{d}}{\mathrm{d}t}(t^6 - 2t^4 + t^2) = 6t^5 - 8t^3 + 2t$.
Chain Rule: From the previous example, we have $\frac{\mathrm{d}}{\mathrm{d}t} f(x(t) - t) = f'(x(t) - t)[x(t) - t]' = f'(x(t) - t)(x'(t) - 1)$.
First, we have $f'(x(t) - t) = 2(x(t) - t) = 2(t^3 - t)$.
Next, we have $x'(t) - 1 = (t^3)' - 1 = 3t^2 - 1$.
Then, $\frac{\mathrm{d}}{\mathrm{d}t} f(x(t) - t) = 2(t^3 - t)(3t^2 - 1) = 6t^5 - 8t^3 + 2t$.

Now, we shall proceed to use this method to solve first order linear PDEs (Partial Differential Equations) below.

First Order Linear PDEs - Classification.

Definition 16. Let the unknown function $u : \mathbb{R}^n \to \mathbb{R}$.

(i) We say that a PDE in u is of **first order** if there exists a function F such that

$$F(u, u_{x_1}, ..., u_{x_n}) = 0. \tag{33}$$

Here, u_{x_i} refers to partial derivatives with respect to the x_i variable. Intuitively, it just means that it contains terms up to its first derivatives only.

(ii) Let us denote the corresponding differential operator for the PDE by $\mathcal{L}$.[a] We say that the PDE is **linear** if $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$ and $\mathcal{L}(au) = a\mathcal{L}(u)$ for all functions u, v, and $a \in \mathbb{R}$.

[a] i.e Given an equation with partial derivatives of u, we write it as $\mathcal{L}u = 0$, where $\mathcal{L}$ is the corresponding differential operator.

Example 17. (Examples.)

(i) Let $u(x, y)$ (ie be a function of two variables), and $\mathcal{L}(u) = u_x + u_y$. We can check see that $\mathcal{L}(u) = 0$ is a first order PDE (since it only contains up to terms in its first derivatives). Furthermore, we can check that the differential operator $\mathcal{L}$ is linear as follows:

- Let u, v be two arbitrary functions. Then, we see that

$$\begin{aligned}\mathcal{L}(u+v) &= (u+v)_x + (u+v)_y \\ &= u_x + v_x + u_y + v_y \\ &= (u_x + u_y) + (v_x + v_y) \\ &= \mathcal{L}(u) + \mathcal{L}(v).\end{aligned} \tag{34}$$

- Let u be an arbitrary function and $a \in \mathbb{R}$ be an arbitrary constant. Then,

$$\begin{aligned}\mathcal{L}(au) &= (au)_x + (au)_y \\ &= au_x + au_y \\ &= a(u_x + u_y) \\ &= a(\mathcal{L}(u) + \mathcal{L}(v)).\end{aligned} \tag{35}$$

(ii) Let $u(x, y)$ (ie be a function of two variables), and $\mathcal{L}(u) = (u_x)^2$. One can check that the corresponding differential operator $\mathcal{L}$ is not linear, since

$$\begin{aligned}\mathcal{L}(au) &= ((au)_x)^2 \\ &= a^2(u_x)^2 \\ &= a^2\mathcal{L}(u)\end{aligned} \tag{36}$$

(ie violates the scaling property).
A rigorous proof of this is as follows. To show that $\mathcal{L}$ is not linear, we just have to show that either of the properties fail. In particular, we want to show that the scaling property fails. That is, it is not true that for all function $u(x, y)$ and $a \in \mathbb{R}$, $\mathcal{L}(au) = a\mathcal{L}(u)$. This is equivalent to cooking up a counter-example; a function $u(x, y)$ and a constant a such that the above equality does not hold. Now, pick $a = 2$ and $u(x, y) = x$. We can then check that $\mathcal{L}(2x) = (2(x)_x)^2 = 4$ while $2\mathcal{L}(x) = 2((x)_x)^2 = 2$, which are clearly not the same!

Solving First Order Linear PDEs.

Method 1: Geometric Method. Given a first order linear PDE, for an unknown function $u(x, y)$ of two variables, of the form

$$a(x,y)u_x + b(x,y)u_y = 0, \tag{37}$$

one can view this as

$$\begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix} \cdot \nabla u(x,y) = 0. \tag{38}$$

Normalizing the vector on the left to be of length one, we have

$$\frac{1}{\sqrt{a^2(x,y)+b^2(x,y)}} \begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix} \cdot \nabla u(x,y) = \mathbf{n} \cdot \nabla u(x,y) = \frac{\partial u}{\partial n} = 0. \tag{39}$$

Thus, at every point (x,y) in the corresponding domain, the directional derivative of u along $\mathbf{n} = \frac{1}{\sqrt{a^2(x,y)+b^2(x,y)}} \begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix}$ (or just $\begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix}$ since the magnitude does not affect the direction) vanishes. This implies that if we sketch out a vector field for $\mathbf{n}$, the function $u(x,y)$ is constant along a curve traced out by $\mathbf{n}$.

Example 18. (Method of Characteristics - Geometric Method). Let $u(x,y)$ be a function of two variables. For the following PDE

$$u_x + u_y = 0, \tag{40}$$

(i) Find a general solution to the PDE above, and

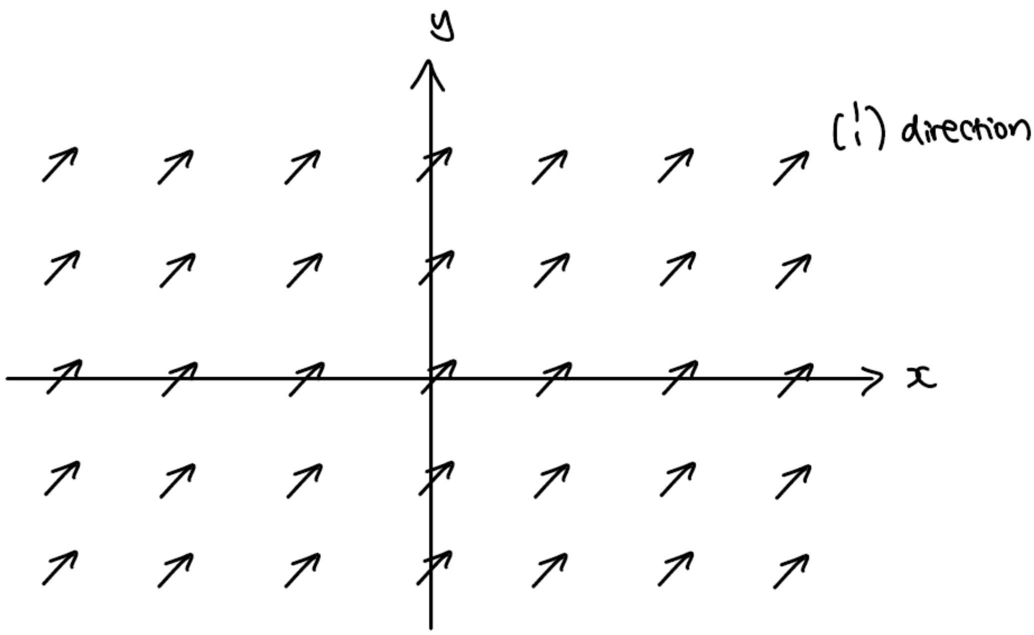
(ii) Sketch a quiver plot showing the tangent directions of the characteristics lines.

Suggested Solution: Rewrite the PDE as

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \nabla u = 0. \tag{41}$$

Thus, u remains constant along lines $y = x + C$ for some constant C. Since $C = y - x$ is a constant, this implies that starting from a $y-$intercept C, u takes the same value along the line $y = x + C$. Thus, $u(x,y) = f(C) = f(y-x)$ for an arbitrary function f.

Quiver Plot:



Remark: I am personally not a fan of this method. I will be focusing on the alternative method, which is more robust, and will be elaborated below.

Method 2: Method of Characteristics.

Given a first order linear PDE, for an unknown function $u(x, y)$ of two variables, of the form

$$a(x, y)u_x + b(x, y)u_y = 0. \tag{42}$$

Suppose that x and y are functions of some common parameter s. (Intuitively, this is the parameter along characteristics curves.) This implies that we have $u(s) = u(x(s), y(s))$ (ie we can view u as a function just depending on s). Then, note that by chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}u(x(s), y(s)) = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s}. \tag{43}$$

Since we are searching for a solution, we shall search for a solution in which $\frac{\mathrm{d}x}{\mathrm{d}s} = a(x(s), y(s))$ and $\frac{\mathrm{d}y}{\mathrm{d}s} = b(x(s), y(s))$. This implies from (43) that

$$\frac{\mathrm{d}}{\mathrm{d}s}u(x(s), y(s)) = a(x, y)u_x + b(x, y)u_y = 0. \tag{44}$$

This implies that u is constant along characteristics curves (described by a common parameter s, ie $(x(s), y(s))$). By solving the following system of ODEs:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}s} = a(x(s), y(s)) \\ \frac{\mathrm{d}y}{\mathrm{d}s} = b(x(s), y(s)) \end{cases} \tag{45}$$

we can determine the functions $x(s)$ and $y(s)$. Setting $x(0)$ and $y(0)$ to be points along the boundary (in which we have imposed certain boundary conditions), then for $s \in \mathbb{R}$, we have

$$u(x(s), y(s)) = u(x(0), y(0)) \tag{46}$$

since u is constant along characteristics. Next, from the obtain functions $x(s)$ and $y(s)$, which depends on $x(0)$ and $y(0)$, we solve $x(0)$ and $y(0)$ as a function of x and s, and plug them into (46) to obtain our solution.

Such an idea can be generalized to $u : \mathbb{R}^n \to \mathbb{R}$ for $n \in \mathbb{N}_{\geq 3}$ (ie higher dimensions), by solving a system of n ODEs analogous to (45). (See Homework 2 Problem 5(a) for an example of this.)

This will be clearer if we look at an explicit example.

Example 19. Solve the following first order PDE using method of characteristics

$$xu_x - u_y = 0, \quad u(x,0) = x^3 \text{ for all } x \in \mathbb{R}, \quad \text{for } x, y \in \mathbb{R}. \tag{47}$$

Suggested Solution:

1. Let $u(x,y) = u(x(s), y(s))$.
2. Compute $\frac{\mathrm{d}}{\mathrm{d}s}u(x(s),y(s)) = u_x \frac{\mathrm{d}x}{\mathrm{d}s} + u_y \frac{\mathrm{d}y}{\mathrm{d}s}$ by chain rule.
3. Set $\frac{\mathrm{d}x}{\mathrm{d}s} = x$ and $\frac{\mathrm{d}y}{\mathrm{d}s} = -1$. This implies that $\frac{\mathrm{d}}{\mathrm{d}s}u(x(s),y(s)) = xu_x - u_y = 0$, ie u is constant along characteristics described by $(x(s), y(s))$.
4. Solve the system of ODEs:
$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}s} = x(s) \\ \frac{\mathrm{d}y}{\mathrm{d}s} = -1 \end{cases} \tag{48}$$
For the first equation, by separation of variables, we can obtain
$$x(s) = x(0)e^s. \tag{49}$$
The second ODE reads:
$$\frac{\mathrm{d}y}{\mathrm{d}s} = -1. \tag{50}$$
Direct integration yields:
$$y(s) = y(0) - s. \tag{51}$$
5. As we would like to start our characteristic curves from the boundary of the domain $y = 0$ in which a boundary condition can be applied to, we set $y(0) = 0$ and $x(0) = x_0$ for some x_0 to be determined.
6. Since u is constant along characteristics, we have
$$u(x(s), y(s)) = u(x(0), y(0)) = u(x_0, 0) = x_0^3. \tag{52}$$
7. Now, given a point (x, y) in the domain, we would like to find out which value of x_0 we should pick such that we arrive at $(x(s), y(s))$ when the parameter takes the value s. This is equivalent to solving (49) and (51) for x_0 as a function of x and y.
(The idea is that x and y are given, and you want to find an s and x_0 such that you can arrive at (x, y) along the characteristics at s, starting from x_0.)
Note that s is yet another "unknown". Given that we have two equations, we can then eliminate the "unknown".
8. (49) and (51) reads
$$\begin{cases} x = x_0 e^s \\ y = -s. \end{cases} \tag{53}$$
Solving for x_0 in terms of x and y, we obtain
$$x_0 = xe^{-s} = xe^y. \tag{54}$$
9. Summarizing all that we have, by substituting (52) to (54), the solution is given by
$$u(x,y) = (xe^y)^3 = x^3 e^{3y}. \tag{55}$$

Remark: A simpler method in the lecture works if we were to just parametrize x in terms of y. ie write

$$u(x,y) = u(x(y), y).$$

The corresponding total derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}y}u(x,y) = \frac{\mathrm{d}}{\mathrm{d}y}u(x(y),y) = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}y} + \frac{\partial u}{\partial y} = u_x\frac{\mathrm{d}x}{\mathrm{d}y} + u_y. \tag{56}$$

Multiplying by -1 to the PDE, we get

$$u_y - xu_x = 0 \tag{57}$$

and we could have picked $\frac{\mathrm{d}x}{\mathrm{d}y} = -x$. However, such a method works more in general if we have PDE of the form

$$a(x,y)u_x + b(x,y)u_y = 0. \tag{58}$$

(ie the original method is more restrictive in the sense that the coefficient of u_y must be 1.) If one would like to consider dividing by $a(x,y)$ throughout the equation, one would have to split solving the characteristics for cases in which $a(x,y) = 0$ and solve in the individual domain separated by "curves" in which $a(x,y) = 0$. This is possibly more complicated than the above method. This makes the above method the most robust method dealing with PDE of the form in (58).

The following is a variant of the method of characteristics presented in Strauss's book, which might have an easier set of computations to do be done to solve the corresponding first order PDE. We shall apply it to solve a slight variant of the PDE above.

Example 20. Solve the following first order PDE using method of characteristics

$$yu_x + u_y = 0, \quad u(x,0) = x^3 \text{ for all } x \in \mathbb{R}, \quad \text{for } x,y \in \mathbb{R}. \tag{59}$$

First, note that if $y = 0$, then $u(x,0)$ is just as given above. Thus, we shall consider a given $(x,y) \in \mathbb{R}^2$ such that $y \neq 0$. From the PDE, this implies that

$$u_x + \frac{1}{y}u_y = 0. \tag{60}$$

Now, by writing $u(x,y) = u(x,y(x))$, the total derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x,y(x)) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{61}$$

by the PDE if we set

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y}. \tag{62}$$

Solving this yields

$$\frac{y^2}{2} = x + C \implies y^2 - 2x = C' \tag{63}$$

for some constants[a] C and C'. Since $\frac{\mathrm{d}u}{\mathrm{d}x}$ is constant along characteristics, we must have $u(x,y(x)) = u(C)$ if we connect the characteristic curves to some point along the given auxiliary condition.
This implies that we must have

$$u(x,y) = f(y^2 - 2x) \tag{64}$$

for some function f to be determined. Using $u(x,0) = x^3$, we have

$$\begin{aligned} u(x,0) = f(0 - 2x) &= x^3 \\ f(-2x) &= -\frac{1}{8}(-2x)^3 \\ f(s) &= -\frac{1}{8}s^3. \end{aligned} \tag{65}$$

Thus, we have

$$u(x,y) = f(y - x^2) = -\frac{1}{8}(y - x^2)^3. \tag{66}$$

[a] Here, we absorb 2 into C and write it as C' for convenience.

One should weigh the advantages of each of these method, try to understand how each of these methods work, and use them accordingly!

3 Discussion 3.

All functions have sufficient smoothness as required, unless stated otherwise.

Additional Considerations for Method of Characteristics.

We might face potential problems when we are trying to solve a PDE by the method of characteristics. These includes

- Multiple values at a given point. This can happen if there exists **two (possibly) different** characteristics with different values of (constant) u along these characteristics connecting to the given point.
- No possible value at a given point. This happens if there are **no** characteristics from the auxiliary boundary conditions that connects to the given point.

The following example illustrates such an idea.

Example 21. Consider the PDE consisting of the unknown function $u(x, y)$ given by

$$u_x + 3x^2 u_y = 0, \tag{67}$$

with boundary conditions

$$\begin{cases} u(x, 0) = x^3 & \text{for all } x \geq 0 \\ u(0, y) = y^2 & \text{for all } y \geq 0. \end{cases} \tag{68}$$

Solve for $u(x, y)$ for $x, y > 0$.

Suggested Solution: Recall that we can solve this by parametrizing the characteristic curves. Note that for both methods as covered in the previous discussion supplement, they are kind of equivalent. We shall display both methods of solving this PDE below.

Method 1 (Strauss's Idea):
Parametrizing the curve by $(x, y) = (x, y(x))$, we write down the corresponding advective derivative for $u(x, y(x))$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}x} u(x, y(x)) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{69}$$

by the PDE if we set

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2. \tag{70}$$

Integrating, we have

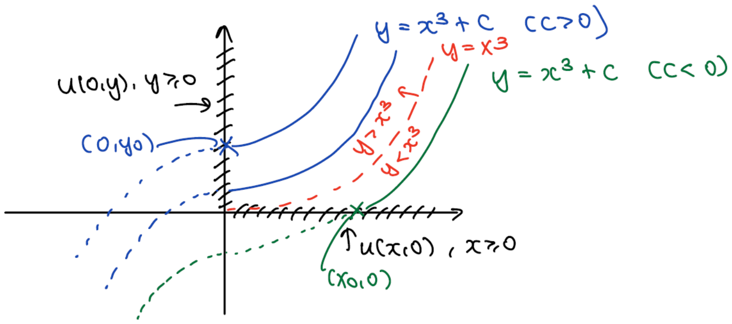
$$y = x^3 + C. \tag{71}$$

This implies that $C = y - x^3$. Thus, we have

$$u(x, y(x)) = u(C) = f(y - x^3) \tag{72}$$

for some undetermined function f.[a]

To make sense of the above equation, the idea is to determine the function f. Thus, we shall consider the shape of the characteristics curves $y = x^3 + C$ for each value of C as shown below.



Thus, from the diagram above, one can see that if $y - x^3 > 0$, then such a point (x, y) is reached by a characteristic curve with $C > 0$ that originates from the $y-$axis. Let this point on the $y-$axis be $(0, y_0)$. Substitute this to (71), we have

$$y_0 = C. \tag{73}$$

Since u is constant along characteristics, this implies that

$$u(x, y(x)) = u(0, y_0) = y_0^2 = C^2 = (y - x^3)^2. \tag{74}$$

On the other hand, if the given point (x, y) is such that $y - x^3 < 0$, then such a point (x, y) is reached by a characteristic curve with $C < 0$. Since our domain is such that $x, y \geq 0$, the characteristic curves must originate from the $x-$axis (since its $y-$intercept is negative). Let $(x_0, 0)$ be this point on the $x-$axis. Substitute this into (71), we have

$$0 = x_0^3 + C \tag{75}$$

so that

$$x_0 = (-C)^{\frac{1}{3}}. \tag{76}$$

Thus, along characteristics, we get

$$u(x, y(x)) = u(x_0, 0) = x_0^3 = \left((-C)^{\frac{1}{3}}\right)^3 = -C = -(y - x^3) = x^3 - y. \tag{77}$$

Hence, we have

$$u(x, y) = \begin{cases} (y - x^3)^2 & \text{for} \quad y > x^3 \\ -(y - x^3) & \text{for} \quad y < x^3. \end{cases} \tag{78}$$

Note that for $y = x^3$, this can be solved using either boundary conditions, with the characteristic curve given by $y = x^3$, and $u = 0$ (since $u(0, 0) = 0$ by either boundary conditions), a constant, along this curve. Alternatively, this can be deduced by continuity of u along $y = x^3$.

Method 2 (Parametrized Curves): Consider the parametrized characteristic curve given by $(x(s), y(s))$. Writing down the advective derivative, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} u(x(s), y(s)) = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} = 0 \tag{79}$$

by the PDE if we set

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}s} = 1 \\ \frac{\mathrm{d}y}{\mathrm{d}s} = 3x^2. \end{cases} \tag{80}$$

Next, we shall solve the ODE system above. Since x also depends on s, it is more instructive to solve the first ODE in x. Integrating directly, we have

$$x(s) = x(0) + s. \tag{81}$$

Plugging this into the second equation, we have

$$\frac{\mathrm{d}y}{\mathrm{d}s} = 3(x(0) + s)^2 \tag{82}$$

and integrating, we obtain

$$y(s) = y(0) + (x(0) + s)^3. \tag{83}$$

Note that from this equation, we immediately have $y(0) = y(0) + (x(0))^3$, and thus $x(0) = 0$. Thus, suppose that $x(0) = 0$ necessarily (ie we are starting our characteristics curves from the $y-$axis). Then we have $x(s) = s$, and thus in the second equation, we get

$$y = y(0) + x^3. \tag{84}$$

Note that since our boundary condition is on the $y-$axis for $y(0) \geq 0$, we are looking at cubic curves $y = x^3$ with non-negative $y-$intercept. Thus, given any x, y, we can solve for $y(0)$ to obtain

$$y(0) = y - x^3. \tag{85}$$

Since $y(0) \geq 0$, this method only works for given x, y such that $y \geq x^3$. Nonetheless, since u is constant on characteristics, we have

$$u(x,y) = u(x(s), y(s)) = u(x(0), y(0)) = u(0, y(0)) = y(0)^2 = (y - x^3)^2 \tag{86}$$

for $y \geq x^3$.

If instead, we want to start from the $x-$axis for $x \geq 0$, since we are required to have $x(0) = 0$, we want to pick a value of $s = s'$ such that $y(s') = 0$. Thus, from (83), we have

$$0 = y(0) + s^3 \tag{87}$$

or $s' = (-y(0))^{\frac{1}{3}}$. From (81), this implies that

$$x(s') = s' = (-y(0))^{\frac{1}{3}}. \tag{88}$$

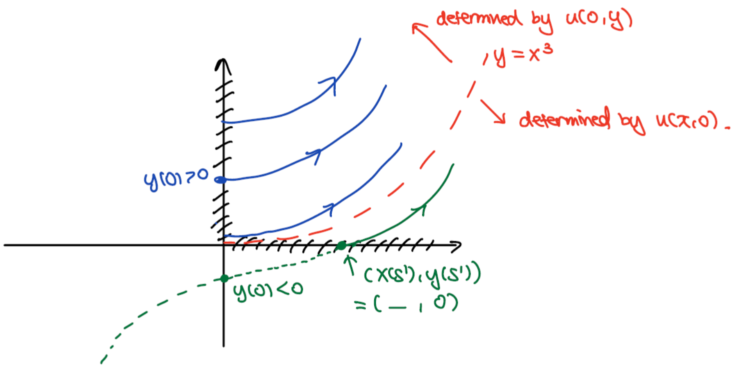
Thus, since u is along the characteristics, we have

$$u(x,y) = u(x(s), y(s)) = u(x(s'), y(s')) = u(x(s'), 0) = (x(s'))^3 = -y(0) = -(y - x^3). \tag{89}$$

Note that this is only valid for $x(s') \geq 0$, ie $-(y - x^3) \geq 0$ or $y \leq x^3$. If $y \leq x^3$, then by (85), we have $y(0) \leq 0$ which is consistent with the diagram below.[b] Thus, we obtain the solution for the entire domain $x, y > 0$ as follows:

$$u(x,y) = \begin{cases} (y - x^3)^2 & \text{for} \quad y \geq x^3 \\ -(y - x^3) & \text{for} \quad y \leq x^3. \end{cases} \tag{90}$$

The following diagram summarizes the computations for Method 2.



[a]Note that the meaning of setting $u(x, y(x)) = u(C)$ is such that we are saying that since u is constant along characteristics, we must be able to follow along characteristics from the given point $(x, y(x))$ to a boundary point (ie in which we impose boundary conditions on). For every (x, y) in the domain, we can find a C and a corresponding point on the boundary such that there exists a characteristic curve that connects from that boundary point to (x, y).

[b]Intuitively, for $y(0) \leq 0$, we instead evolve along characteristic curves such that we instead start on the $x-$axis and use the boundary conditions there!

Remark: If we did not impose boundary conditions such that say $u(x, 0)$ for $x \geq 0$ (along the y-axis), then there are no characteristic curves originating from our boundary conditions that can reach any (x, y) such that $y > x^3$! (See both methods for a physical interpretation/visualization for this).

Another Remark (Question raised by someone in class): It has been pointed out that the u obtained is not necessarily C^1 (i.e. not differentiable across the boundary $y = x^3$), although its certainly continuous in the domain. However, we also note that if there is a solution, this must be the solution. For this class, we shall assume that the solutions obtained from formal computations are indeed the solutions. The lack of differentiability is addressed in what we call a "weak solution", which is way outside of the scope of this class.

Mathematical Modelling and PDEs.

Example 22. (Conservation Laws.) Suppose we have PDE for $u(x,t)$ of the form:

$$u_t + (f(u))_x = 0, \tag{91}$$

where $t > 0$ and $x \in (a,b)$ for some $a < b \in \mathbb{R}$, with $f : \mathbb{R} \to \mathbb{R}$ being a C^1 function. Suppose that we impose the following **Dirichlet boundary conditions**[a]:

$$u(a,t) = u(b,t) = 0 \tag{92}$$

for all time $t \geq 0$.
Let us denote the quantity $M(t)$ by

$$M(t) = \int_a^b u(x,t)\mathrm{d}x. \tag{93}$$

Prove that

$$M(t) = M(0) \tag{94}$$

for all $t > 0$.

Suggested Solution. First, we compute $\frac{\mathrm{d}M}{\mathrm{d}t}$ as follows:[b]

$$\begin{aligned}\frac{\mathrm{d}M}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x \\ &= \int_a^b \partial_t u(x,t)\mathrm{d}x \quad \text{by Leibinz Rule} \\ &= -\int_a^b (f(u))_x(x,t)\mathrm{d}x \quad \text{by the PDE} \\ &= -(f(u(b,t)) - f(u(a,t))) \quad \text{by Fundamental Theorem of Calculus} \\ &= -(f(0) - f(0)) \quad \text{applying Dirichlet boundary conditions} \\ &= 0.\end{aligned} \tag{95}$$

This implies that M does not change with time, and thus (94) is true. Alternatively, one can take integration with respect to time on both sides of the equation to obtain (85).

An interpretation of this example is that if we think of u as mass, and $f(u)$ as the term representing the flux of mass across the boundary, then we see that the total mass $M = \int_a^b u(x,t)\mathrm{d}x$ mass is conserved. This is consistent with our interpretation of the transport/continuity equation if we set $q(x,t) = f(u(x,t))$ as the flux. See here or Lecture 6 materials for more information.

[a] Recap: This means that the value of the function at the boundary (in which here, it is at $x = a$ and $x = b$), is zero.
[b] Note that Leibinz Rule here is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} u(x,t)\mathrm{d}x = \int_{a(t)}^{b(t)} \partial_t u(x,t)\mathrm{d}x + u(x,b(t))\frac{\mathrm{d}b(t)}{\mathrm{d}t} - u(x,a(t))\frac{\mathrm{d}a(t)}{\mathrm{d}t}.$$

See here for more information.
Further Note: For the analysts in class, interchanging $\frac{\mathrm{d}}{\mathrm{d}t}$ with $\int \mathrm{d}x$ requires justification. In particular, for our case in which a and b does not depend on t, using your favorite Analysis textbook (ie Rudin/Tao)/materials from 131B, we note that this is allowed as long as $\partial_t u$ exists and $\int_a^b \int_0^\infty |\partial_t u|\mathrm{d}t\mathrm{d}x$ is bounded (so that Fubini's theorem applies). A stronger condition as mentioned in the attached link is to just require the continuity of u and ∂tu on a neighborhood of $[a,b] \times \{t\}$. However, since at the start of all these discussion supplements, we assume that our function u has the required regularity whenever required, then we just assume that this is indeed the case!

Common types of boundary conditions:

- Dirichlet Boundary Conditions: $u|_{\partial\Omega} = 0$ (ie value on the boundary is 0).
- Neumann Boundary Conditions: $\frac{\partial u}{\partial n}|_{\partial\Omega} = 0$ (ie derivative in some direction (usually normal to the domain) along on the boundary is 0). This is used to model the fact that u does not change at the boundary of the domain of interest (ie heat not flowing across the boundary).

Example 23. (Deriving a Physical Equation.) (Strauss Exercise 1.3.5.) Derive the equation of one-dimensional diffusion in a medium that is moving along the x axis to the right at constant speed V.

Suggested Solution: Following the derivation in Strauss, we first let $u(x,t)$ be the concentration (mass per unit length) of an object of interest (say a dye or something).
Remark: I'm following Strauss' derivation because its a little simpler for this case and straightforward.

Consider a section of the medium from x_0 to x_1. The total mass of the dye is given by

$$M(t) = \int_{x_0}^{x_1} u(x,t)\mathrm{d}x. \tag{96}$$

This implies that

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_{x_0}^{x_1} u_t(x,t)\mathrm{d}x. \tag{97}$$

By Fick's law, we know that the flow rate to the right is proportional to $-u_x$. Thus, we have

$$\begin{aligned}\text{Mass flow rate, } \frac{\mathrm{d}M_1}{\mathrm{d}t}(x,t) &= \text{Inflow (to the right) at } x_0 + \text{Inflow (to the left) at } x_1 \\ &= \text{Flow to the right at } x_0 - \text{Flow to the right at } x_1 \\ &= -Du_x(x_0,t) - (-Du_x(x_1,t)) \\ &= Du_x(x_1,t) - Du_x(x_0,t)\end{aligned} \tag{98}$$

Here, the convention is that $\mathrm{d}M$ refers to mass flowing into the interval (x_0, x_1). (ie Convention - Use physical laws such that we compute the amount that flows into the medium - if it is negative, then the actual flow is out of the medium!)

Furthermore, the motion of the medium carrying the dye along with it contributes to

$$\text{Moving Medium Contribution, } \frac{\mathrm{d}M_2}{\mathrm{d}t}(x,t) = -Vu(x_1,t) + Vu(x_0,t) \tag{99}$$

This can be seen from a simple physical idea:

$$\begin{aligned}\text{Influx rate from left end, } x_0 &= \text{Increase in mass/time at } x_0 \\ &= \text{Mass/Small length section near } x_0 \times \text{Small length section/time} \\ &= \text{Concentration near } x_0 \times \text{ Speed} = +u(x_0,t) \times V.\end{aligned} \tag{100}$$

Here, we attribute the $+$ sign from the fact that the moving medium towards the right drags mass into the section of interest. Equating the sum of the two contributions to the change in mass in (97), we have (with D as the constant of proportionality)

$$\int_{x_0}^{x_1} u_t(x,t)\mathrm{d}x = Du_x(x_1,t) - Du_x(x_0,t) - Vu(x_1,t) + Vu(x_0,t). \tag{101}$$

Taking ∂_{x_1}, we get

$$u_t(x_1,t) = Du_{xx}(x_1,t) - Vu_x(x_1,t). \tag{102}$$

Since x_1 is arbitrary, we have

$$u_t(x,t) = Du_{xx}(x,t) - Vu_x(x,t). \tag{103}$$

Types of 2nd order linear PDEs. Let us consider the general form of a second order linear PDE below:

$$a_{11}(x,y)u_{xx} + 2a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} + a_1(x,y)u_x + a_2(x,y)u_y + a_0(x,y)u = 0. \tag{104}$$

Note that the factor 2 is introduced for convenience.

(i) We say that the PDE is **elliptic** at (x,y) if $a_{12}^2(x,y) < a_{11}(x,y)a_{22}(x,y)$.

(ii) We say that the PDE is **hyperbolic** at (x,y) if $a_{12}^2(x,y) > a_{11}(x,y)a_{22}(x,y)$.

(iii) We say that the PDE is **parabolic** at (x,y) if $a_{12}^2(x,y) = a_{11}(x,y)a_{22}(x,y)$.

Note that one way to visualize the criterion is to think of a corresponding matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} = a_{12} & a_{22} \end{bmatrix} \tag{105}$$

that is symmetric, and $\det(A) > 0$,< 0, and $= 0$ implies that the PDE is elliptic, hyperbolic, and parabolic respectively.

Furthermore,

(i) If the PDE is **elliptic**, by a change of coordinates, we can reduce it to $u_{xx} + Au_{yy} + ... = 0$. If the terms in ... is 0 and $A = 1$, then
$$u_{xx} + u_{yy} = 0$$
is the **Laplace's equation**.

(ii) If the PDE is **hyperbolic**, by a change of coordinates, we can reduce it to $u_{xx} - Au_{yy} + ... = 0$. If the terms in ... is 0 and $A = 1$, then
$$u_{xx} - u_{yy} = 0$$
is the **wave equation**. (Treat $x \to t$ and $y \to x$.)

(iii) If the PDE is **parabolic**, by a change of coordinates, we can reduce it to $u_{xx} + 0u_{yy} - Au_y + ... = 0$. (Since the coefficient of u_{yy} vanishes, we look at the next highest order term.) If the terms in ... is 0, then
$$u_{xx} - Au_y = 0$$
is the **diffusion/heat equation.** (Treat $y \to t$.)

Example 24. Find the regions in the xy plane where the equation

$$yu_{xx} - 2u_{xy} + xu_{yy} = 0 \tag{106}$$

is elliptic, hyperbolic, or parabolic.

Suggested Solution: The corresponding matrix $A = \begin{bmatrix} y & -1 \\ -1 & x \end{bmatrix}$ has determinant $= yx - 1$. Thus, it is

- Elliptic if $yx - 1 > 0$ ie $xy > 1$,
- Hyperbolic if $xy < 1$ and
- Parabolic if $xy = 1$.

4 Discussion 4.

All functions have sufficient smoothness as required, unless stated otherwise.

(1−dimensional) Wave Equation.

Consider the wave equation with boundary conditions [3] given by

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R}\times\{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{107}$$

As seen in the lectures, the solution to (107), known as the d'Alembert's solution[4], is given by

$$\boxed{u(x,t) = \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\mathrm{d}y.} \tag{108}$$

Note that it does not matter if you pick c to be the positive or the negative root - the solution in (108) would be the same (you should try it out!). Nonetheless, for convenience and physical sake, we shall pick c to be positive.

Example 25. Using d'Alembert's solution (which you can quote, without rederiving it), solve the wave equation:

$$\begin{cases} u_{tt}(x,t) = 16u_{xx}(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = e^{-x^2} & \text{on } \mathbb{R}\times\{t=0\}, \\ u_t(x,0) = x & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{109}$$

Suggested Solution: Using d'Alembert's solution as in (108) (by substituting $\phi(x) = e^{-x^2}$ and $\psi(x) = x$), we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} y \,\mathrm{d}y \\ &= \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right) + \frac{1}{4c}y^2\Big|_{y=x-ct}^{y=x+ct} \\ &= \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right) + \frac{1}{4c}\left((x+ct)^2 - (x-ct)^2\right). \end{aligned} \tag{110}$$

Since $c^2 = 16$, we have $c = 4$. Thus, we have

$$u(x,t) = \frac{1}{2}\left(e^{-(x-4t)^2} + e^{-(x+4t)^2}\right) + \frac{1}{16}\left((x+4t)^2 - (x-4t)^2\right). \tag{111}$$

[3] Here, "in $\mathbb{R}\times(0,\infty)$" means $(x,t)\in\mathbb{R}\times(0,\infty)$, ie $x\in\mathbb{R}$ and $t>0$.
Similarly, "on $\mathbb{R}\times\{t=0\}$" means $(x,t)\in\mathbb{R}\times\{t=0\}$, i.e., for all $x\in\mathbb{R}$ but only at $t=0$.

[4] Remark: From d'Alembert's solution, it is possible that we impose initial data ϕ and ψ that are not sufficiently smooth (since we are imposing that u is equals to a combination of ϕ and ψ to some extent). For an example, see the three pluck problem in Exercise 3 of Homework 4, where $\phi(x) = 1-|x|$ for $|x|<1$. Here, ϕ is not differentiable at $x=\pm 1$. However, following a remark from Discussion 3, we shall accept this as the formal solution, which constitutes to a "weak" solution that is way outside of the scope of this class! Feel free to take graduate classes in applied/pure PDEs to learn more these solutions! References: 251ABC, 266ABC.

Following the derivation for d'Alembert's solution, this involves factorizing the differential operator for the wave equation. For instance, consider the wave equation on $\mathbb{R}$ given by

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, \infty). \tag{112}$$

We factorize the differential operator on the left as follows:

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right). \tag{113}$$

We then deduce by a change of variable that the general form of the solution to (112) is given by

$$u(x,t) = f(x - ct) + g(x + ct), \tag{114}$$

where f and g are arbitrary functions (of sufficient regularity), determined using initial conditions. These can be viewed as a sum of travelling waves, moving at speed c in the positive and the negative x direction! One can re-derive (114) by looking at the corresponding factorization of the differential operator, in which will be covered in the example on the next page.

Example 26. Derive sum-of-travelling wave solutions (similar to (114)) to the following PDE:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x \partial t} - 2\frac{\partial^2 u}{\partial x^2} = 0. \tag{115}$$

Suggested Solution: The factorization problem

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x \partial t} - 2\frac{\partial^2}{\partial x^2}\right) = \left(\frac{\partial}{\partial t} + A\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + B\frac{\partial}{\partial x}\right) \tag{116}$$

can be viewed as factorizing a second-order polynomial in t and x as follows:

$$t^2 - xt - 2x^2 = (t + Ax)(t + Bx). \tag{117}$$

(View $t \to \frac{\partial}{\partial t}$ and $x \to \frac{\partial}{\partial x}$.) Since we can write

$$t^2 - xt - 2x^2 = (t + x)(t - 2x), \tag{118}$$

then

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x \partial t} - 2\frac{\partial^2}{\partial x^2}\right) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}\right). \tag{119}$$

Now, let $\xi = t - x$ and $\eta = 2t + x$ (swap the coefficients on t and x and put a negative sign on one of them - the intuition comes from method of characteristics (see below)).
Then, see that for any arbitrary smooth function f, we have by (a careful application of) chain rule,

$$\begin{aligned}
\frac{\partial}{\partial x} f(x,t) &= \frac{\partial}{\partial x} f(\xi(x,t), \eta(x,t)) \\
&= \frac{\partial f(\xi,\eta)}{\partial \xi}\frac{\partial \xi(x,t)}{\partial x} + \frac{\partial f(\xi,\eta)}{\partial \eta}\frac{\partial \eta(x,t)}{\partial x} \\
&= \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right) f(\xi,\eta) \\
&= \left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) f(\xi,\eta).
\end{aligned} \tag{120}$$

Similarly, we have

$$\begin{aligned}
\frac{\partial}{\partial t} f(x,t) &= \frac{\partial}{\partial t} f(\xi(x,t), \eta(x,t)) \\
&= \frac{\partial f(\xi,\eta)}{\partial \xi}\frac{\partial \xi(x,t)}{\partial t} + \frac{\partial f(\xi,\eta)}{\partial \eta}\frac{\partial \eta(x,t)}{\partial t} \\
&= \left(\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\right) f(\xi,\eta) \\
&= \left(\frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta}\right) f(\xi,\eta).
\end{aligned} \tag{121}$$

Now, we check that $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$ (given by (120) + (121)) gives

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) f(x,t) = 3\frac{\partial}{\partial \eta} f(\xi, \eta). \tag{122}$$

Furthermore, we can also check that $\left(\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}\right)$ (given by (121) -2 (120)) gives

$$\left(\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}\right) f(x,t) = 3\frac{\partial}{\partial \xi} f(\xi, \eta). \tag{123}$$

Thus, (119) reduces to (by picking $f = u$)

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}\right) u(x,t) = 9\frac{\partial}{\partial \eta}\frac{\partial}{\partial \xi} u(\xi, \eta) = 0 \tag{124}$$

where the last equation is by the given PDE (115). Taking partial integration with respect to η (and dropping the factor 9 by dividing it on both sides), we get

$$u_\xi(\xi, \eta) = f_1(\xi) \tag{125}$$

for arbitrary function f_1. Then, perform partial integration with respect to ξ to obtain

$$u(\xi, \eta) = f_2(\xi) + g(\eta) \tag{126}$$

where $f_2(\xi) = \int^\xi f_1(s)\mathrm{d}s$, and g is an arbitrary function. Substitute the subsitutents back to obtain

$$u(x,t) = f(x-t) + g(x+2t) \tag{127}$$

for arbitrary functions f and g (here we set $f(\xi) = f_2(-\xi)$).

Remark: Note that in (119), we have the operator in the form

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}\right). \tag{128}$$

The intuition to pick the substituents $\xi = t - x$ and $\eta = t + 2x$ is as follows. If we would like to solve a PDE of the form

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u(x,t) = 0, \tag{129}$$

we can employ the method of characteristics [5] to deduce that

$$u(x,t) = f(x-t) \tag{130}$$

for arbitrary function f. One can repeat a similar argument for the other operator to obtain $x + 2t = C$!

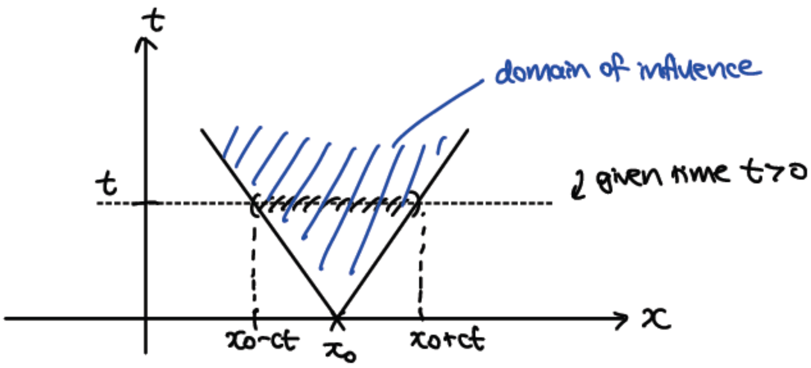
[5] This is done by solving the corresponding ODE to obtain $x = t + C$ so we have $C = x - t$.

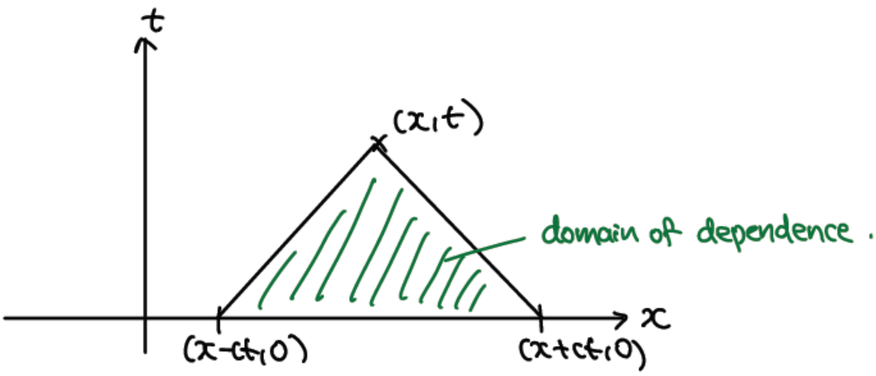
Yet another important concept is the domain of dependence and influence. The idea originates from the d'Alembert's solution. Recall that we have for a wave equation $u_{tt} - c^2 u_{xx} = 0$ for $x \in \mathbb{R}, t > 0$, we have

$$\boxed{u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\mathrm{d}y.} \tag{131}$$

Recall that it is easy to check that $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$. For $t > 0$, we can analyze the contribution of each terms as follows:

- The first term corresponds to an initial waveform $\phi(x)$ which splits into two pieces with half the amplitude, and one goes in the positive x direction at speed c (represented by $\frac{1}{2}\phi(x-ct)$), while the other goes in the negative x direction at speed c (represented by $\frac{1}{2}\phi(x+ct)$)[6]. This implies that the solution depends on ϕ at $x+ct$ and $x-ct$.
- The second term is a little more subtle (at least for me). Nonetheless, it has the physical structure that given a point (x,t), the solution depends on ψ on the interval $[x-ct, x+ct]$.
- In totality, we have that the solution at (x,t) depends on initial data along the interval $[x-ct, x+ct]$. This is what we call the **domain of dependence**.
- In reverse, if we ask that at a given point along the initial data (say $(x_0, 0)$), and ask which points on the spacetime diagram it would affect, this constitutes to an outward expanding cone. At every given $t > 0$, the physical domain in which it affects is given by $[x_0 - ct, x_0 + ct]$. This is what we call the **domain of influence**.





These physical interpretation can help us to solve problems in a "physical" sense instead of computing a bunch of really complicated dependencies using the d'Alembert's solution directly. The following example illustrates this.

[6] Recall that $\phi(x) \to \phi(x+a)$ refers to translation of the graph of $\phi(x)$ in the **negative** direction by a units.

Example 27. Consider the dynamics of a sqaure wave propagated by the wave equation with $c = 1$, as shown below:

$$u_{tt} = u_{xx} \quad \text{in } \mathbb{R} \times (0, \infty). \tag{132}$$

with

$$\phi(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{133}$$

and $\psi(x) = 0$ for all $x \in \mathbb{R}$.

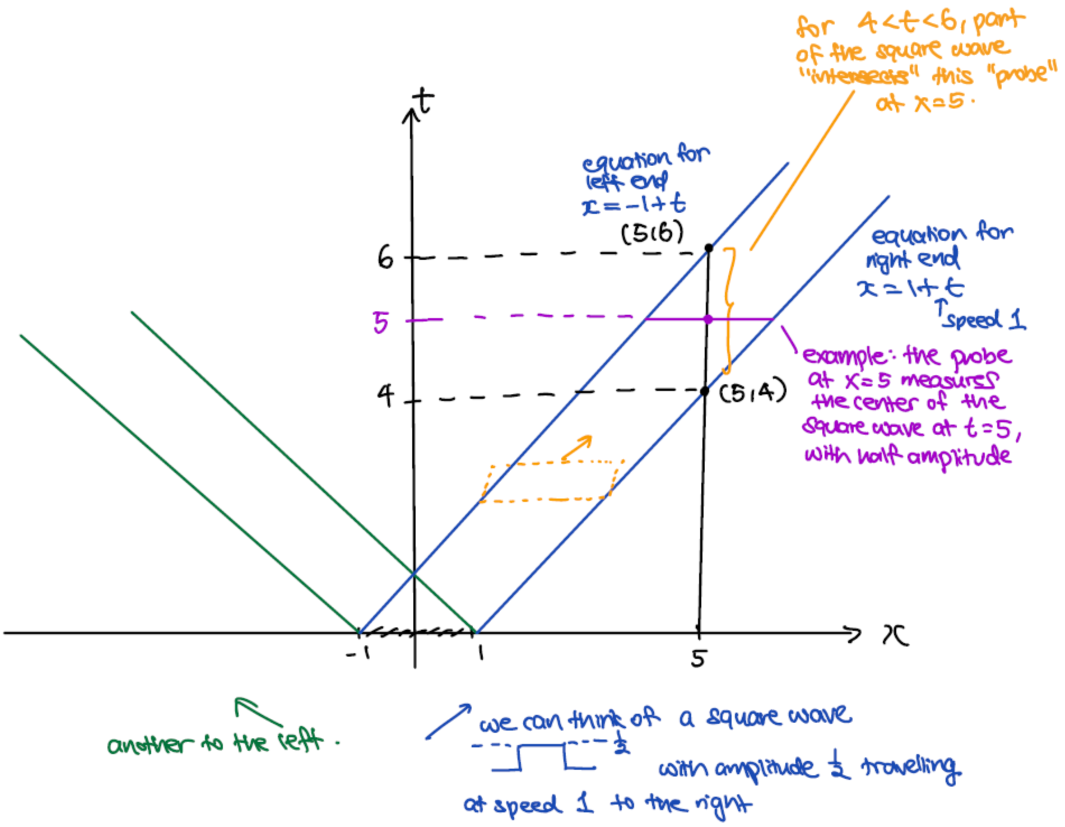
Consider an observer at $x = 5$.

(i) Determine the time interval for which $u(5, t) \neq 0$.

(ii) Determine the value of $u(5, t)$ for t in the time interval determined in (i).

Suggested Solution:

Instead of using the explicit d'Alembert's solution, we shall do this question mainly with the use of a spacetime diagram! By the physical interpretation in the previous page, the initial waveform created by ϕ splits into two, and travels in the opposite direction with speed $c = 1$. The following diagram summarizes all the arguments that we need. It also follows from what is mentioned in the diagram that the solution to (i) and (ii) are $t \in (4, 6)$ and $u(5, t) = \frac{1}{2}$ for $t \in (4, 6)$ respectively.



Remarks: One can use a similar argument for Homework 4 Exercise 3, though one should be careful that the plucked string does not have a uniform amplitude. In Exercise 4, it might be better to use d'Alembert's formula to evaluate the explicit solution $u(x, t)$ as ψ rather than ϕ is given, though a spacetime plot and interpreting the integral physically (see Lecture 8) might help!

Energy Method and Application to Uniqueness of Solutions to PDEs.

As a recap, let us consider the wave equation, given by

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, \infty). \tag{134}$$

By physical considerations (kinetic + potential - see Lecture 10 for more details), we should consider the following energy function given by

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}(u_t)^2(x,t) + \frac{c}{2}(u_x)^2(x,t)\mathrm{d}x. \tag{135}$$

First, we have the following property:

> **Proposition 28.** If $E(t) < +\infty$ for all $t \geq 0$, then $E(t)$ in (135) is constant with respect to t.

As per usual, the trick is to take $\frac{\mathrm{d}}{\mathrm{d}t}$ on (135) and show that it is equals to 0. Thus, we compute $\frac{\mathrm{d}}{\mathrm{d}t}E(t)$ as follows:

$$\begin{aligned}\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \frac{1}{2}(u_t)^2(x,t) + \frac{c^2}{2}(u_x)^2(x,t)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\frac{1}{2}(u_t)^2(x,t) + \frac{c^2}{2}(u_x)^2(x,t)\right)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} (u_t u_{tt} + c^2 u_x u_{xt})(x,t)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} (u_t u_{tt} + c^2 u_x u_{tx})(x,t)\mathrm{d}x\end{aligned} \tag{136}$$

where the last equality $u_{xt} = u_{tx}$ follows from Schwarz theorem (ie interchanging second order partial derivatives if u is sufficiently smooth).
Next, we employ integration by parts to shift ∂_x from u_{tx} to u_x to obtain

$$\begin{aligned}\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \int_{-\infty}^{\infty} (u_t u_{tt} + c^2 u_x u_{tx})(x,t)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} (u_t u_{tt} - c^2 u_{xx} u_t)(x,t)\mathrm{d}x + c^2(u_x u_t)(x,t)\Big|_{x=-\infty}^{x=+\infty} \\ &= \int_{-\infty}^{\infty} (u_t)(u_{tt} - c^2 u_{xx})(x,t)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} 0 \,\mathrm{d}x = 0.\end{aligned} \tag{137}$$

The last equality is by the wave equation itself, while the boundary terms u_x and u_t vanishes. Since we assume that u is at least C^2 (so that the wave equation makes sense), this implies that u_t and u_x are continuous. Thus, for $E(t) = \int_{-\infty}^{\infty} \frac{1}{2}(u_t)^2(x,t) + \frac{c}{2}(u_x)^2(x,t)\mathrm{d}x < +\infty$ and that u_x and u_t are (uniformly) continuous[7], then it is necessary that $u_x, u_t \to 0$ as $x \to \pm\infty$.

An application of the above is to prove the uniqueness of solutions to the wave equation with general boundary conditions as given in (107), repeated below:

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t = 0\}, \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{138}$$

[7]Uniform continuity is also required for this argument to work. This can be incorporated in the required regularity of unknown functions to be solved. An alternative way to justify this is by using d'Alembert's solution to show that the solution is compactly supported (ie vanishes outside of a closed interval) for all given time $t > 0$, but this would require an additional assumption that the initial data ϕ and ψ are compactly supported. One can refer to Strauss to see how such a justification was done.

If doing Analysis concerns you, just let your intuition overwhelm you and assume that this is true since it is intuitively true (hiding the required analysis under "sufficient regularity")!

Suppose that u_1 and u_2 are solutions to (138), then we have

$$\begin{cases} (u_1)_{tt}(x,t) - c^2(u_1)_{xx}(x,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ (u_1)(x,0) = \phi(x) & \text{on } \mathbb{R}\times\{t=0\}, \\ (u_1)_t(x,0) = \psi(x) & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{139}$$

and

$$\begin{cases} (u_2)_{tt}(x,t) - c^2(u_2)_{xx}(x,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ (u_2)(x,0) = \phi(x) & \text{on } \mathbb{R}\times\{t=0\}, \\ (u_2)_t(x,0) = \psi(x) & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{140}$$

Taking (140) - (139), we have

$$\begin{cases} w_{tt}(x,t) - c^2 w_{xx}(x,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ w(x,0) = 0 & \text{on } \mathbb{R}\times\{t=0\}, \\ w_t(x,0) = 0 & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{141}$$

with $w := u_2 - u_1$. Now, define the energy given by

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}(w_t)^2(x,t) + \frac{c}{2}(w_x)^2(x,t)\mathrm{d}x. \tag{142}$$

Next, we make the following observations:

- $E(0) = \int_{-\infty}^{\infty} \frac{1}{2}(w_t)^2(0,t) + \frac{c}{2}(w_x)^2(0,t)\mathrm{d}x = 0$ by the given conditions in (141).[8]
- $E(t) = E(0)$ by the Proposition above (E is constant with respect to time).
- Furthermore, $E(t) \geq 0$ for all $t \geq 0$ (see the explicit expression in (142), in which each term in the integral is squared, so they must be non-negative). [9]

Combining all three facts, we have that $0 = E(0) = E(t) \geq 0$ and thus $E(t) = 0$ for all $t \geq 0$. This implies that for all $t \geq 0$, we have

$$\int_{-\infty}^{\infty} \frac{1}{2}(w_t)^2(x,t) + \frac{c}{2}(w_x)^2(x,t)\mathrm{d}x = 0. \tag{143}$$

Since w_t and w_x are continuous, and $(w_t)^2$ and $(w_x)^2 \geq 0$, then we must have $w_t = w_x = 0$ for all $x \in \mathbb{R}, t \geq 0$.[10] Integrating (partially) with respect to t from $w_t = 0$, we obtain that

$$w(x,t) = f(x). \tag{144}$$

Take $\frac{\partial}{\partial x}$ on both sides and use the fact that $w_x = 0$ to obtain

$$0 = w_x(x,t) = f'(x) = 0. \tag{145}$$

Solve $f'(x) = 0$ to obtain $f(x) = C$ (a constant independent of x and t!). Thus, we deduce that

$$w(x,t) = C \tag{146}$$

for all $x \in \mathbb{R}$, $t \geq 0$. Use the initial condition $w(x,0) = 0$ to deduce that $C = 0$. Then, since $w = u_2 - u_1$, we deduce that

$$u_2(x,t) = u_1(x,t) \tag{147}$$

for all $x \in \mathbb{R}$ and $t \geq 0$. This implies that the solution to the wave equation (i.e (138)) is unique!

Remark: Most uniqueness arguments using the energy method follows an extremely similar argument as the one above, so it is extremely important to know the proof of it (which explains why I've included them here).

[8] $w_t = 0$ is obtained directly. For $w_x = 0$, we obtain this from $w(x,0) = 0$ and take $\frac{\partial}{\partial x}$ on both sides of the equation.

[9] For this case in which $E(t) = E(0)$, we really did not have to check that $E(t) \geq 0$. However, for physical reasons and more often than not, we get that $E(t) \leq E(0)$ rather than equality. Thus, as explained (towards the end of this page), we have to check $E(t) \geq 0$ for all time t.

[10] To be more rigorous, this is basically a result from Analysis, which requires continuity of the function(s) that is/are squared. Strauss calls it the "First Vanishing Theorem". The interested analysts in the class can refer to Appendix A in Strauss for the proof of this!

Another Remark: Note that when we made the set of observations above, **it suffices to show that** $E(t)$ **is non-increasing** ("decreasing"; not necessarily constant), while we still need $E(0) = 0$ and $E(t) \geq 0$. This yields $0 = E(0) \geq E(t) \geq 0$ (note that the second mathematical operator is now $\geq$ instead of $=$), and deduce that $E(t) = 0$ for all $t \geq 0$. See Homework 4 Exercise 5(b).

Yet Another Remark: Thus, if we would like to construct the energy function E that works, we would need it to satisfy the "observations" as stated above. In particular, the third condition $E(t) \geq 0$ restricts our choice of integrals to consist of squared terms, which explains why such an energy used (ie kinetic/potential energy, which constitutes to squared terms) works really well!

Example 29. Consider the "wave equation" given by

$$\begin{cases} (u_{tt} - u_{xt} - 2u_{xx})(x,t) = 0 & \text{in } (0,1)\times(0,\infty), \\ u(0,t) = u(1,t) = 0 & \text{for } t \geq 0, \\ u(x,0) = u_t(x,0) = 0 & \text{on } [0,1]\times\{t=0\}. \end{cases} \tag{148}$$

By considering the energy

$$E(t) = \frac{1}{2}\int_0^1 (u_t)^2(x,t) + 2(u_x)^2(x,t)\mathrm{d}x, \tag{149}$$

show that if u is smooth, then $u(x,t) = 0$ for all $x \in [0,1], t \geq 0$.

Suggested Solution:

- $E(t) \geq 0$ follows from the fact that both $(u_t)^2$ and $(u_x)^2$ are ≥ 0.
- $E(0) = 0$ since $u_t(x,0) = 0$ and $u(x,0) = 0$ implies (by taking ∂_x) that $u_x(x,0) = 0$.

It remains to show that $\frac{\mathrm{d}E(t)}{\mathrm{d}t} \geq 0$ (thus E is either constant or non-decreasing).

Compute

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u_t)^2(x,t) + 2(u_x)^2(x,t)\mathrm{d}x \\ &= \int_0^1 (u_t u_{tt} + 2u_x u_{xt})(x,t)\mathrm{d}x \\ &= \int_0^1 (u_t u_{tt} + 2u_x u_{tx})(x,t)\mathrm{d}x \quad \text{by Schwarz Theorem} \\ &= \int_0^1 (u_t u_{tt} - 2u_{xx}u_t)(x,t)\mathrm{d}x + 2u_x u_t|_{x=0}^{x=1} \quad \text{using Integrating by parts} \\ &= \int_0^1 u_t(u_{tt} - 2u_{xx})(x,t)\mathrm{d}x \quad \text{by boundary conditions} \\ &= \int_0^1 (u_t u_{xt})(x,t)\mathrm{d}x \quad \text{by PDE} \\ &= \int_0^1 (u_t u_{tx})(x,t)\mathrm{d}x \quad \text{by Schwarz} \\ &= \int_0^1 \left(\frac{(u_t)^2}{2}\right)_x (x,t)\mathrm{d}x \quad \text{by black magic/reverse chain rule} \\ &= \frac{(u_t)^2}{2}(1,t) - \frac{(u_t)^2}{2}(0,t) \quad \text{by Fundamental theorem of Calculus} \\ &= 0 \quad \text{by boundary conditions.} \end{aligned} \tag{150}$$

Note that the boundary terms vanish because $u_t(1,t) = u_t(0,t) = 0$ (take ∂_t on $u(0,t) = u(1,t) = 0$). With the three conditions checked, we then deduce that $0 \leq E(t) \leq E(0) = 0$ and thus $E(t) = 0$ for all $t \geq 0$. This implies that $u_t = u_x = 0$ for all $x \in [0,1], t \geq 0$. Following a similar argument as on the previous page, we deduce that $u(x,t) = C$ for some constant independent of x and t, and thus $u(x,t) = 0$ for $x \in (0,1)$ and $t \geq 0$ by applying any of the boundary conditions.

5 Discussion 5.

All functions have sufficient smoothness as required, unless stated otherwise.

Example 30. Consider the following PDE:

$$e^y u_x + e^x u_y = 0 \tag{151}$$

on $(x, y) \in (0, \infty) \times (0, \infty)$, with boundary conditions:

$$u(x, 0) = x^3 \quad \text{for all } x \geq 0 \tag{152}$$

and

$$u(0, y) = y^2 \quad \text{for all } y \geq 0. \tag{153}$$

Solve for $u(x, y)$ in the domain $(0, \infty) \times (0, \infty)$.

Suggested Solution: Since e^y does not vanish, we can divide the PDE by that throughout to obtain

$$u_x + e^{x-y} u_y = 0. \tag{154}$$

Paramterize u with $u(x, y) = u(x, y(x))$. The advective derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} u(x, y(x)) = u_x(x, y(x)) + \frac{\mathrm{d}y}{\mathrm{d}x} u_y(x, y(x)) = 0 \tag{155}$$

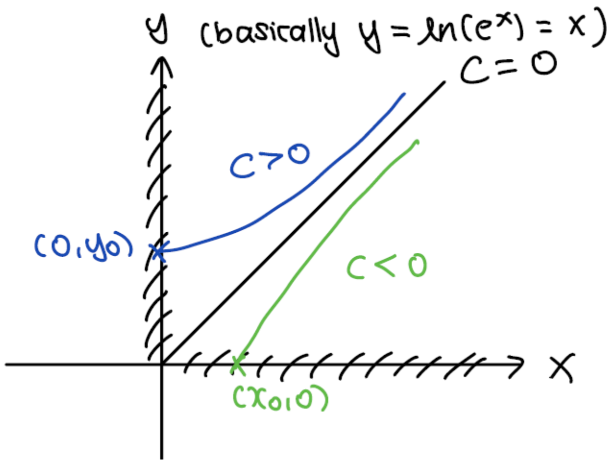
if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x-y}. \tag{156}$$

Solving this ODE by separation of variables, see that we have

$$\begin{aligned} \int e^y \mathrm{d}y &= \int e^x \mathrm{d}x \\ e^y &= e^x + C \\ y &= \ln\left(e^x + C\right) \text{ and} \\ C &= e^y - e^x \end{aligned} \tag{157}$$

where the $\ln$ term is understood to exist if x takes a corresponding value such that $e^x + C > 0$. What do the characteristic lines look like?



$C > 0$.

Now, consider the case for $C > 0$ and let $(0, y_0)$ be point on the $y-$axis in which the characteristic curve for a given $C > 0$ lies on. Thus, we can solve for this point as follows:

$$y_0 = \ln\left(e^0 + C\right) = \ln(1 + C) > 0. \tag{158}$$

Thus, given any (x, y) with $y > x$, (ie above the line $y = x$), we then have

$$u(x, y(x)) = u(0, y_0) = y_0^2 = \ln(1 + C)^2 = (\ln(1 + e^y - e^x))^2 . \tag{159}$$

<u>$C < 0$.</u>
Now, consider the case for $C < 0$ and let $(x_0, 0)$ be point on the $x-$axis in which the characteristic curve for a given $C < 0$ lies on. Thus, we can solve for this point as follows:

$$0 = \ln(e_0^x + C) \to x_0 = \ln(1 - C) > 0. \tag{160}$$

Thus, given any (x, y) with $y < x$, (ie below the line $y = x$), we then have

$$u(x, y(x)) = u(x_0, 0) = x_0^3 = \ln(1 - C)^3 = (\ln(1 + e^x - e^y))^3 . \tag{161}$$

<u>$C = 0$.</u>
Since along the characteristic curve (ie $y = x$), we can trace all the way back to the origin, we then have

$$u(x, y(x)) = u(0, 0) = 0. \tag{162}$$

Example 31. Let us consider an observer in $3D$, with the observer's position represented in Cartesian coordinates (x, y, z), and the observer's path parametrized by time t (that is, $\mathbf{r}(t) = (x(t), y(t), z(t))$). With respect to a stationary observer, the density of matter $u(x, y, z, t)$ is described by the PDE:

$$u_t + u_x + tu_y + t^2 u_z = 0 \tag{163}$$

for $t > 0$. Assume that we have an initial distribution of matter at $t = 0$, given by $u(x, y, z, 0) = f(x, y, z)$ for some smooth function f on $\mathbb{R}^3$.

(i) Given a position $(x(t), y(t), z(t))$ of the observer at time t, write down the velocity vector $\mathbf{r}'(t)$ describing the velocity of the observer such that the density of matter does not change relative to this observer.

(ii) Solve the PDE (163) for $u(x, y, z, t)$, with $(x, y, z, t) \in \mathbb{R}^3 \times (0, \infty)$.

Suggested Solution:

(i) Invoking the advective derivative for $(x(t), y(t), z(t), t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} u(x(t), y(t), z(t), t) = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial u}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial u}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = 0 \tag{164}$$

if

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= 1 \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= t \\ \frac{\mathrm{d}z(t)}{\mathrm{d}t} &= t^2. \end{cases} \tag{165}$$

Thus, the corresponding velocity vector is given by $\mathbf{r}'(t) = (x'(t), y'(t), z'(t)) = (1, t, t^2)$ at a given time t.

(ii) Let us try to solve for $x(t), y(t)$ and $z(t)$ in (165). By directly integrating the system in (165), we obtain

$$\begin{cases} x(t) &= x(0) + t \\ y(t) &= y(0) + \frac{t^2}{2} \\ z(t) &= z(0) + \frac{t^3}{3}. \end{cases} \tag{166}$$

so we have

$$\begin{cases} x(0) &= x(t) - t \\ y(0) &= y(t) - \frac{t^2}{2} \\ z(0) &= z(t) - \frac{t^3}{3}. \end{cases} \tag{167}$$

Thus, along characteristics for which u is constant, we have

$$u(x(t), y(t), z(t), t) = u(x(0), y(0), z(0), 0) = f(x(0), y(0), z(0)) = f\left(x - t, y - \frac{t^2}{2}, z - \frac{t^3}{3}\right) \tag{168}$$

as the required solution.

Example 32. Let us consider a similar set up as in Homework 3 Exercise 2, in which we consider the 1D flow of heat along a bar. Assume that the bar lies along $0 \leq x \leq L$. We shall summarize the physical scenario below:

- Amount of thermal energy in unit length of bar is related to its temperature T by $C_P T$, where C_P is the constant representing heat capacity.
- Thermal energy is conducted along the bar, and the rate of flow of thermal energy at a point x is given by Fick's law; $q(x,t) = -k\frac{\partial T}{\partial x}$, where k represents the thermal conductivity.

The PDE satisfied by the temperature $T(x,t)$ along the rod at position $x \in (0, L)$ at an arbitrary time $t > 0$ is given by

$$T_t = \frac{C_P}{k} T_{xx}. \tag{169}$$

You can assume that both C_p and k are positive constants.
Write down the corresponding PDE with the appropriate boundary conditions satisfied by T if the following changes were made (simultaneously)

1. The temperature at $x = L$ is maintained at some temperature T_0.
2. Thermal energy is supplied on the left end of the bar at $x = 0$ at a rate of Q.

Suggested Solution: Note that the PDE remains unchanged (we do not have a "uniform" heat source that increases the temperature of the rod throughout, and thus, we do not have a source term s as in Homework 3 Problem 1. Nonetheless, we have to write down the corresponding boundary conditions satisfied by T as follows.

1. This implies that $T(L,t) = T_0$ for all $t \geq 0$, since the temperature is fixed at this value.
2. By Fick's law, this is basically the value of q at $x = 0$. Thus, we have $Q = q(0,t) = -kT_x(0,t)$, and thus $T_x(0,t) = -\frac{Q}{k}$ for all $t > 0$.

Example 33. A commonly used equation to model traffic flow is known as the Burger's equation. Let $u(x,t)$ be the density of cars at position $x \in [0,L]$ and time $t \geq 0$ (where L is the length of the road of interest). Then, under some sort of non-dimensionalization, u satifies the Burger's equation as given below:

$$u_t(x,t) + u(x,t)u_x(x,t) = 0. \tag{170}$$

Let us denote the total number of cars on the road by $N(t)$, which we can obtained as follows:

$$N(t) = \int_0^L u(x,t)\mathrm{d}x. \tag{171}$$

One possible way to include a physical boundary condition that the number of cars on the road stay constant, is to impose that $\frac{\mathrm{d}N}{\mathrm{d}t} = 0$ for all $t > 0$. However, this might be unnecessary, and one could instead impose

$$u(L,t) = u(0,t) \tag{172}$$

for all $t > 0$.

Show that indeed, if one impose (172), then the number of cars on the road $x \in [0,L]$ stays constant for all time $t > 0$.

Suggested Solution: As per usual, our strategy is to compute $\frac{\mathrm{d}N(t)}{\mathrm{d}t}$. Indeed, we have

$$\begin{aligned}
\frac{\mathrm{d}N(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u(x,t)\mathrm{d}x \\
&= \int_0^L u_t(x,t)\mathrm{d}x && \text{apply PDE next} \\
&= \int_0^L u u_x \, \mathrm{d}x && \text{with } (x,t) \text{ supressed} \\
&= \int_0^L \left(\frac{u^2}{2}\right)_x \mathrm{d}x && \text{by reverse chain rule} \\
&= \frac{u^2}{2}(L,t) - \frac{u^2}{2}(0,t) && \text{by Fundamental Theorem of Calculus} \\
&= 0 && \text{since } u(L,t) = u(0,t) \text{ for all } t > 0.
\end{aligned} \tag{173}$$

Thus, we can then deduce that the number of cars on the road stays constant for all time $t > 0$.

Example 34. Consider the wave equation:

$$\begin{cases} u_{tt}(x,t) = 16u_{xx}(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = \begin{cases} x^2 - 255 & \text{for } |x| \le 16 \\ 0 & \text{otherwise} \end{cases} & \text{on } \mathbb{R}\times\{t=0\}, \\ u_t(x,0) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{otherwise} \end{cases} & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{174}$$

Compute

(i) $u(21,1)$,

(ii) $u(0,2)$, and

(iii) $u(8104,2022)$.

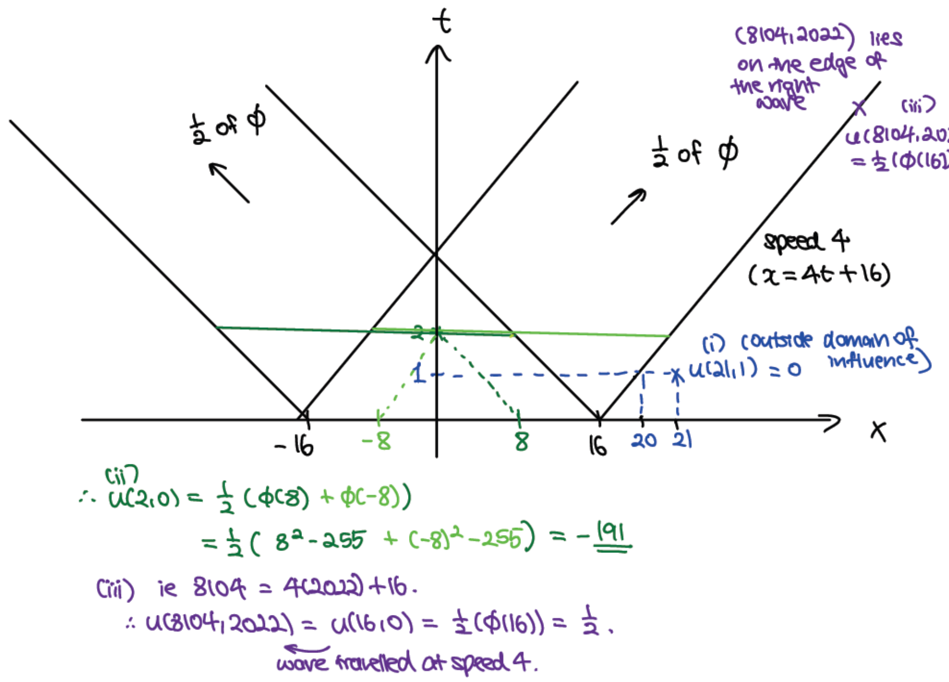
Hint: This is a question that is testing mainly the concept on domain of dependence/influence.

Suggested Solution: The wave speed is given by $c = \sqrt{16} = 4$. By d'Alembert's solution, explicitly given by

$$u(x,t) = \frac{1}{2}(\phi(x-4t) + \phi(x+4t)) + \frac{1}{8}\int_{x-4t}^{x+4t} \psi(y)\mathrm{d}y, \tag{175}$$

we can consider the contribution by ϕ and ψ individually, and then add them up in the end!

$\underline{\phi.}$



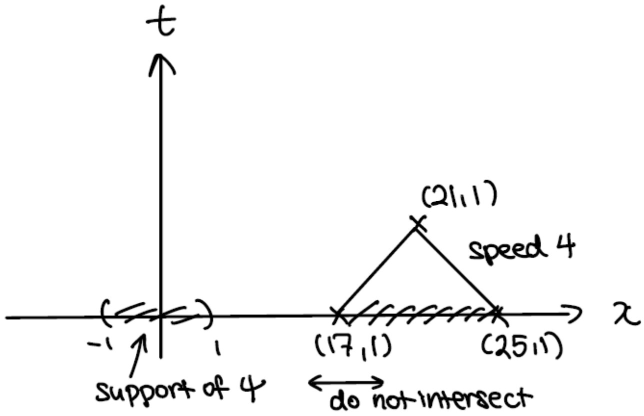
$\underline{\psi.}$ Note that the integrand ψ is 1 if $|y| \le 1$. Thus, we can interpret the integral as $\frac{1}{8}|(x-4t, x+4t)\cap(-1,1)|$ (here $|\cdot|$ refers to the length of the interval). One can interpret this geometrically, or just notice that since we are given a bunch of points, we can just treat this as a bunch of computational exercise. Hence, we have

- At $(21,1)$, we have $\frac{1}{8}|(21-4, 21+4)\cap(-1,1)| = 0$.
- At $(0,2)$, we have $\frac{1}{8}|(-8,8)\cap(-1,1)| = \frac{2}{8} = \frac{1}{4}$.

- At $(8104, 2022)$, we have

$$\frac{1}{8}\left|(8104 - 4\times 2022, 8104 + 4\times 2022)\cap(-1,1)\right| = \frac{1}{8}\left|(16, 16192)\cap(-1,1)\right| = 0.$$

(Note that this is consistent with the concept of domain of dependence, which can be understood as follows. For a given t, and a point x, we look an interval given by $(x-4t, x+4t)$ in which u depends on. This can be seen as a cone drawn backwards with speed 4 and to see if this coincides with the support[a] of the initial data ψ (which is $(-1,1)$). If they do not intersect, then ψ did not impact this point (x,t)! See diagram below for a visualization of this, which helps you to visualize what you are computing!)



Summarizing, we then have

- $u(21,1) = 0 + 0 = 0$,
- $u(0,2) = -191 + \frac{1}{4} = -190.75$, and
- $u(8104, 2022) = \frac{1}{2} + 0 = \frac{1}{2}$.

[a] Support of a function refers to the set of points in the domain of the function in which the function does not vanish.

Example 35. Consider the following modified wave equation:

$$\begin{cases} (u_{tt} + u_t - u_{xx} + u)(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{176}$$

By considering the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2(x,t) + (u_x)^2(x,t) + u^2(x,t)\mathrm{d}x, \tag{177}$$

prove that the above modified wave equation has at most one $u \in C^2$ solution, with u_x and u_t vanishing at $x \to \pm\infty$.

Suggested Solution: Suppose that u_1 and u_2 are solutions to (176). Then,

$$\begin{cases} ((u_1)_{tt} + (u_1)_t - (u_1)_{xx} + u_1)(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ (u_1)(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t=0\}, \\ (u_1)_t(x,0) = \psi(x) & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{178}$$

and

$$\begin{cases} ((u_2)_{tt} + (u_2)_t - (u_2)_{xx} + u_2)(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ (u_2)(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t=0\}, \\ (u_2)_t(x,0) = \psi(x) & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{179}$$

Take (179) $-$ (178) to obtain the following PDE for $w = u_2 - u_1 \in C^2$ below

$$\begin{cases} (w_{tt} + w_t - w_{xx} + w)(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ w(x,0) = 0 & \text{on } \mathbb{R} \times \{t=0\}, \\ w_t(x,0) = 0 & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{180}$$

Replacing u by w in the energy in (177), we have

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2(x,t) + (w_x)^2(x,t) + w^2(x,t)\mathrm{d}x. \tag{181}$$

One can check that $E(0) = 0$ and $E(t) \geq 0$ for all $t \geq 0$ (see Discussion Supplement 4 for details). It remains to compute $\frac{\mathrm{d}E}{\mathrm{d}t}$ and check that it is ≤ 0 for all $t > 0$. Thus, we have (with (x,t) suppressed)

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} (w_t)^2 + (w_x)^2 + w^2\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}((w_t)^2 + (w_x)^2 + w^2)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} w_t(w_{tt}) + w_x w_{xt} + w w_t \mathrm{d}x \quad \text{by Chain rule} \\ &= \int_{-\infty}^{\infty} w_t(w_{tt}) + w_x w_{tx} + w w_t \mathrm{d}x \quad w \in C^2 \\ &= \int_{-\infty}^{\infty} w_t(w_{tt}) - w_{xx} w_t + w w_t \mathrm{d}x - w_x w_t|_{x=-\infty}^{x=\infty} \quad \text{Integration by parts} \\ &= \int_{-\infty}^{\infty} w_t(w_{tt}) - w_{xx} w_t + w w_t \mathrm{d}x \quad \text{boundary terms vanish} \\ &= \int_{-\infty}^{\infty} w_t(w_{tt} - w_{xx} + w)\mathrm{d}x \quad \text{regrouping terms} \\ &= \int_{-\infty}^{\infty} w_t(-w_t)\mathrm{d}x \quad \text{by PDE} \\ &= -\int_{-\infty}^{\infty} (w_t)^2\mathrm{d}x \leq 0 \quad \text{for all } t > 0. \end{aligned} \tag{182}$$

For a discussion on why boundary terms vanish, see Discussion Supplement 4 (for this case, since $(u_1)_x$, $(u_1)_t$, $(u_2)_x$, and $(u_2)_t$ all vanish at $x \to \pm\infty$, then w_t and w_x do too!). The above three properties thus implies that $w_t = w_x = w = 0$ in $\mathbb{R} \times (0, \infty)$. In particular, we immediately obtain $w = 0$, and thus $u_1(x, t) = u_2(x, t)$ in $\mathbb{R} \times (0, \infty)$.

Additional Exercises

Exercise 36. (Strauss 1.2 Exercise 1) Solve the PDE

$$2u_t + 3u_x = 0 \tag{183}$$

with the boundary condition $u(x, 0) = \sin(x)$ for all $x \in \mathbb{R}$, for $u(x, t)$ with $(x, t) \in \mathbb{R} \times (0, \infty)$.

Exercise 37. (Strauss 1.2 Exercise 3.) Solve the PDE

$$(1 + x^2)u_x + u_y = 0 \tag{184}$$

for its general solution.

Exercise 38. (Strauss 1.2 Exercise 7.)

(i) Solve the PDE

$$yu_x + xu_y = 0, u(0, y) = e^{-y^2}. \tag{185}$$

(ii) In which region of the $xy-$plane is the solution uniquely determined?

Exercise 39. (Strauss 2.1 Exercise 7.) Consider the 1D wave equation

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = \phi(x) & \text{on } \mathbb{R} \times \{t = 0\}, \\ u_t(x, 0) = \psi(x) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{186}$$

Show that if ϕ and ψ are odd functions of x, then the solution $u(x, t)$ is also an odd function in x for all $t > 0$.

Exercise 40. (Strauss 2.1 Exercise 9.) Consider the 1D "wave" equation

$$\begin{cases} (u_{xx} - 3u_{xt} - 4u_{tt})(x,t) = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = x^2 & \text{on } \mathbb{R} \times \{t = 0\}, \\ u_t(x, 0) = e^x & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{187}$$

Solve for $u(x, t)$ in $\mathbb{R} \times (0, \infty)$.

Exercise 41. (Strauss 2.1 Exercise 11.) Find the general solution $u(x, t)$ of

$$3u_{tt} - 10u_{xt} + 3u_{xx} = \sin(x + t) \quad \text{in } \mathbb{R} \times (0, \infty) \tag{188}$$

in $\mathbb{R} \times [0, \infty)$.

Exercise 42. (Strauss 1.6 Exercise 4.) Classify the following second order linear PDE (ie is this elliptic, parabolic or hyperbolic):

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0. \tag{189}$$

Exercise 43. Consider the 1D wave equation

$$\begin{cases} (u_{tt} - u_{xx})(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{otherwise} \end{cases} & \text{on } \mathbb{R} \times \{t = 0\}, \\ u_t(x,0) = 0 & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{190}$$

This is also known as a **dam break problem**, which models the flow of water along a river when a dam breaks and water rushes from behind (with a sudden difference in height u of say 1 at $t = 0$ when the dam breaks completely at that instance).

(i) Determine $u(2,1)$ and

(ii) $u(-2,t)$ as a function of t for $t \geq 0$.

Exercise 44. Consider the **telegrapher's equation** given by

$$u_{tt} + au_t = c^2 u_{xx} \quad \text{in } \mathbb{R} \times (0,\infty) \tag{191}$$

with $a > 0$. This models the vibration of a string under frictional damping. The classical energy function (in the absence of damping) is given by

$$E(t) = \frac{1}{2}\int_0^1 (u_t)^2(x,t) + c^2(u_x)^2(x,t)\mathrm{d}x. \tag{192}$$

In Homework 4 Exercise 5(b), we have shown that the energy decreases monotonically with time. Now, consider instead that $a < 0$. Show that the energy increases monotonically with time. This should be consistent with our intuition that au_t corresponds to a damping term, and thus by reversing the sign of a, we are introducing energy to the system over time!

Exercise 45. Let us consider a similar set up as in Homework 3 Exercise 2, in which we consider the 1D flow of heat along a bar. Assume that the bar lies along $0 \leq x \leq L$. We shall summarize the physical scenario below:

- Amount of thermal energy in unit length of bar is related to its temperature T by $C_P T$, where C_P is the constant representing heat capacity.
- Thermal energy is conducted along the bar, and the rate of flow of thermal energy at a point x is given by Fick's law; $q(x,t) = -k\frac{\partial T}{\partial x}$, where k represents the thermal conductivity.

The PDE satisfied by the temperature $T(x,t)$ along the rod at position $x \in (0,L)$ at an arbitrary time $t > 0$ is given by

$$T_t = \frac{C_P}{k} T_{xx}. \tag{193}$$

You can assume that both C_p and k are positive constants.
Write down the corresponding PDE with the appropriate boundary conditions satisfied by T if the following changes were made (simultaneously)

1. The temperature at $x = 0$ is fixed at an increasing rate, with $T(0,0) = T_0$ and that the temperature at this end increases at a rate of r (ie the temperature increases by r units for every unit increase in time t, here the units are suppressed, and represents the corresponding S.I units).
2. No conduction of the heat happens at $x = L$.

For solutions to the above problems,

- You might be able to find the solutions to the first seven problems from the solution manual of Strauss' book.
- For the third problem from the last, the numerical values are given by 0 and $u(2,t) = \begin{cases} 1 & \text{for } t \in [0,2] \\ \frac{1}{2} & \text{for } t > 2 \end{cases}$.
- The second problem from the last should be doable (basically something similar to Homework 4 Exercise 5(b)). Since it is not due yet (and that it is extremely similar to that), I will not provide the solution to this exercise.
- For the last problem,
  1. This implies that $T(0,t) = T_0 + rt$.
  (We can check that at $t = 0$, we have $T = T_0$, and that $\frac{\mathrm{d}T(0,t)}{\mathrm{d}t} = r$. Alternatively, we can solve this ODE $\frac{\mathrm{d}T(0,t)}{\mathrm{d}t} = r$ with boundary condition $T(0,0) = T_0$.)
  2. $T_x(L,t) = 0$ for all $t > 0$ (this can be understood using Fick's law with $q = 0$).

6 Discussion 6.

All functions have sufficient smoothness as required, unless stated otherwise.

Maximum Principle for $1D$-diffusion equation.

Consider the $1D-$diffusion equation on a **bounded spatial domain** as described mathematically below.

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } [a,b]\times(0,\infty), \\ u(x,0) = \phi(x) & \text{on } [a,b]\times\{t=0\}, \\ u(a,t) = f(t) & \text{for } t\geq 0, \\ u(b,t) = g(t) & \text{for } t\geq 0. \end{cases} \tag{194}$$

Consider solving $u(x,t)$ on a closed bounded subset of $[a,b]\times(0,\infty)$; that is, for $(x,t)\in[a,b]\times[0,T]$. Then, the maximum principle states that the maximum value of $u(x,t)$ is attained on either $x=a$, $x=b$ or $t=0$. Mathematically,

$$\begin{aligned} \max_{(x,t)\in[a,b]\times[0,T]} u(x,t) &= \max_{(x,t)\in[a,b]\times[0,T]}\{\phi(x),f(t),g(t)\} \\ &= \max\left\{\max_{x\in[a,b]}\phi(x),\ \max_{t\in[0,T]}f(t),\ \max_{t\in[0,T]}g(t)\right\}. \end{aligned} \tag{195}$$

Here are some remarks on the maximum principle (relevant for this class):

- The **minimum principle** can be derived from the maximum principle. To see this, apply the maximum principle to $-u$ as follows. By linearity of the given diffusion equation in (194), we have

$$\begin{cases} (-u)_t(x,t) = D(-u)_{xx}(x,t) & \text{in } [a,b]\times(0,\infty), \\ (-u)(x,0) = -\phi(x) & \text{on } [a,b]\times\{t=0\}, \\ (-u)(a,t) = -f(t) & \text{for } t\geq 0, \\ (-u)(b,t) = -g(t) & \text{for } t\geq 0. \end{cases} \tag{196}$$

  Apply maximum principle to u to obtain

$$\begin{aligned} \max_{(x,t)\in[a,b]\times[0,T]} -u(x,t) &= \max\left\{\max_{x\in[a,b]}-\phi(x),\ \max_{t\in[0,T]}-f(t),\ \max_{t\in[0,T]}-g(t)\right\} \\ -\min_{(x,t)\in[a,b]\times[0,T]} u(x,t) &= \max\left\{-\min_{x\in[a,b]}\phi(x), -\min_{t\in[0,T]}f(t), -\min_{t\in[0,T]}g(t)\right\} \\ &= -\min\left\{\min_{x\in[a,b]}\phi(x),\ \min_{t\in[0,T]}f(t),\ \min_{t\in[0,T]}g(t)\right\} \\ \min_{(x,t)\in[a,b]\times[0,T]} u(x,t) &= \min\left\{\min_{x\in[a,b]}\phi(x),\ \min_{t\in[0,T]}f(t),\ \min_{t\in[0,T]}g(t)\right\}. \end{aligned} \tag{197}$$

  Here, we have used some mathematical properties of max and min, mainly, $\max_D(-u) = -\min_D(u)$ for any function u and the corresponding subset of the function's domain, D; and $\max\{-a,-b,-c\} = -\min\{a,b,c\}$ for any real numbers a,b, and c.

- For this class, it is only required for you to know the maximum principle on bounded domains (ie basically, $(x,t)\in[a,b]\times[0,T]$). **If the spatial domain is not bounded, then you can't apply the maximum principle.** Most, if not all, of the problems given would be the $1D-$diffusion equation on bounded domains. You just have to know how to apply **both** the maximum and minimum principle.

- What if the spatial domain (ie in x) is bounded, but the temporal domain (ie in t) is not? Maximum principle still holds, and is given by (modified from (195)) [11]

$$\max_{(x,t)\in[a,b]\times[0,\infty)} u(x,t) = \max\left\{\max_{x\in[a,b]}\phi(x),\ \max_{t\in[0,\infty)}f(t),\ \max_{t\in[0,\infty)}g(t)\right\}. \tag{198}$$

[11] This can be proven looking at a closed bounded domain $[a,b]\times[0,T]$ for an arbitrary T, and noting that the argument is somewhat independent of T. Thus, we can send T to infinity to conclude.
Note: To be mathematically prudent, one should note that if we are taking the max over a domain that is not closed, it is possible that the maximum does not exist on such a domain. (Recall that on a closed subset of $\mathbb{R}$, the maximum of a continuous function exists by Extreme Value Theorem; this is not guaranteed to exist in the same subset of $\mathbb{R}$ if it is not closed.) Thus, if you have taken the 131 series, it suffices to think of max as sup. If not, I would suggest that you don't have to bog yourself down with such details.

- Understanding the proof of the maximum principle is somewhat optional (as seen from Homework 5 that it is actually just an optional problem). For students who are interested in analysis of PDEs, feel free to attempt the optional problem on the homework. The method of attack for that problem would be inspired from the proof of maximum principle as covered in Lecture 11.

Example 46. (Strauss 2.3 Exercise 4 Modified.) Consider the diffusion equation given by

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) & \text{in } [0,1]\times(0,\infty), \\ u(x,0) = 4x(1-x) & \text{on } [0,1]\times\{t=0\}, \\ u(0,t) = 0 & \text{for } t\geq 0, \\ u(1,t) = 0 & \text{for } t \geq 0. \end{cases} \tag{199}$$

Prove that $0 \leq u(x,t) \leq 1$ for all $(x,t) \in [0,1]\times[0,\infty)$.

Suggested Solution:

By the maximum principle, we have

$$\max_{(x,t)\in[0,1]\times[0,\infty)} u(x,t) = \max\left\{\max_{x\in[0,1]} 4x(1-x),\ \max_{t\in[0,\infty)} 0,\ \max_{t\in[0,\infty)} 0\right\}. \tag{200}$$

One can compute $\max_{x\in[0,1]} 4x(1-x)$ using techniques from single variable calculus (Math 31A). Let $\phi(x) = 4x(1-x)$ on $x \in [0,1]$. Then, at the maximum point, we have $\phi'(x) = 4 - 8x = 0$. Thus, $x = \frac{1}{2}$ corresponds to a critical point. One can check that $\phi''(x) = -8 < 0$ so this is a local maximum point. To compute the global maximum, we compare the value of ϕ at the local maximum and at the boundary:

- $\phi\left(\frac{1}{2}\right) = 1$,
- $\phi(0) = 0$, and
- $\phi(1) = 0$.

Thus, $\max_{x\in[0,1]} 4x(1-x) = 1$. As an added bonus, we also have that $\min_{x\in[0,1]} 4x(1-x) = 0$ (ie we just have to compare the values of ϕ at the boundary points and pick the lower value; in this case, they are both 0). Hence, we have

$$\begin{aligned} \max_{(x,t)\in[0,1]\times[0,\infty)} u(x,t) &= \max\left\{\max_{x\in[0,1]} 4x(1-x),\ \max_{t\in[0,\infty)} 0,\ \max_{t\in[0,\infty)} 0\right\} \\ &= \max\{1,0,0\} = 1. \end{aligned} \tag{201}$$

This implies that $u(x,t) \leq 1$ for all $(x,t) \in [0,1]\times[0,\infty)$.

To show that $u(x,t) \geq 0$, we will instead try to apply the minimum principle. This is given by

$$\begin{aligned} \min_{(x,t)\in[0,1]\times[0,\infty)} u(x,t) &= \min\left\{\min_{x\in[0,1]} 4x(1-x),\ \min_{t\in[0,\infty)} 0,\ \min_{t\in[0,\infty)} 0\right\} \\ &= \min\{1,0,0\} = 0. \end{aligned} \tag{202}$$

This immediately implies that $u(x,t) \geq 0$ for all $(x,t) \in [0,1]\times[0,\infty)$ and we are done.

Similarity Solutions.

One possible way to derive the fundamental solution (which we will define in a mathematically rigorous sense a little later) would be via constructing a similarity solution. This idea is inspired from Physics, in which by using simple dimensional analysis, one can "guess" the form in which the solution should take to construct a solution to the given PDE. Instead of outlining the general steps required, it is easier to look at an example as shown below.

Example 47. A heated plate lies along the positive $x-$axis, and air flows over the plate in the $x-$direction. There is a simple shear flow above the plate. Assume that the plate surface is at a temperature T_0. Thus, for $x > 0$ and $y > 0$, the temperature field $T(x, y)$ above the plate is given by

$$\gamma y T_x(x, y) = D T_{yy}(x, y). \tag{203}$$

Here, γ (the shear rate), and D (diffusivity), and T_0 are constants. Assume that at $x = 0$ the air is at ambient temperature $T(0, y) = 0$, while on the surface of the plate, we have $T(x, 0) = T_0$ and far from the plate, we have $T(x, \infty) = 0$.

(i) Find a functional form to the PDE of the form $T(x, y) = a(x) f\left(\frac{y}{L(x)}\right)$. Here, finding the functional form means explicitly finding $a(x)$ and $L(x)$, and deriving an ODE for $f(\eta)$ with $\eta = \frac{y}{L(x)}$.

(ii) By deriving the corresponding boundary conditions for the ODE and solving the ODE for f, determine the similarity solution $T(x, y)$ to the above PDE for $x, y \geq 0$.

Suggested Solution:

Understanding the requirements of the question.

When we would like to find a similarity solution of the form $T(x, y) = a(x) f\left(\frac{y}{L(x)}\right)$, note that we are usually assuming that $\eta = \frac{y}{L(x)}$ is **unitless/dimensionless**, so η represents a dimensionless variable. This also implies that $f\left(\eta = \frac{y}{L(x)}\right)$ is a function of a dimensionless variable, and thus dimensionless! Here, we denote $[X]$ to mean the "units of X".
One should note that although physically, x and y are spatial coordinates for this question and represents the same physical unit (of length), we shall assume for generalizability that they are "different".

Step 1: Determine the functional form of $L(x)$ by determining the units of y in terms of x and other physical constants.

This is because as mentioned, $\eta = \frac{y}{L(x)}$ is unitless, so $[y] = [L]$. Thus, to determine the units of L, it suffices to determine the units of y.

From the PDE, we have

$$\begin{aligned} [\gamma y T_x] &= [D T_{yy}] \\ \frac{[\gamma][y][T]}{[x]} &= \frac{[D][T]}{[y]^2} \\ [y] &= \left(\frac{[D][x]}{[\gamma]}\right)^{\frac{1}{3}}. \end{aligned} \tag{204}$$

This implies that $[L] = \left(\frac{[D][x]}{[\gamma]}\right)^{\frac{1}{3}}$, and thus, it makes physical sense to postulate

$$L(x) = \left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}. \tag{205}$$

Step 2: Use the auxiliary (initial/boundary/conservation of mass) condition to determine the functional form of $a(x)$.

For this question, we are given the boundary condition $T(x,0) = T_0$. Substitute this into the form of our similiary solution, we have

$$T(x,0) = a(x)f\left(\frac{0}{L(x)}\right) = a(x)f(0) = T_0 \tag{206}$$

Without loss of generality[a], we set $f(0) = 1$, and deduce that $a(x) = T_0$ for all $x \geq 0$.

Step 3: Derive the corresponding ODE for f.
Reaping our achievements in the previous two steps, we have

$$T(x,y) = T_0 f\left(\frac{y}{\left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}}\right) = T_0 f(\eta) \tag{207}$$

with $\eta(x,y) = \frac{y}{\left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}} = y\gamma^{\frac{1}{3}}D^{-\frac{1}{3}}x^{-\frac{1}{3}}$. Now, we have to compute T_x and T_{yy} using Chain rule, as described below:

- $\frac{\partial}{\partial x}T(x,y) = T_0\frac{\partial}{\partial x}f(\eta(x,y)) = T_0 f'(\eta(x,y))\frac{\partial \eta(x,y)}{\partial x} = -\frac{1}{3}T_0 f'(\eta(x,y))y\gamma^{\frac{1}{3}}D^{-\frac{1}{3}}x^{-\frac{4}{3}}$.
- $\frac{\partial}{\partial y}T(x,y) = T_0\frac{\partial}{\partial y}f(\eta(x,y)) = T_0 f'(\eta(x,y))\frac{\partial \eta(x,y)}{\partial y} = T_0 f'(\eta(x,y)).\gamma^{\frac{1}{3}}D^{-\frac{1}{3}}x^{-\frac{1}{3}}$
- $$\begin{aligned}\frac{\partial^2}{\partial y^2}T(x,y) &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}T(x,y)\right) = \frac{\partial}{\partial y}(T_0 f'(\eta(x,y))\gamma^{\frac{1}{3}}D^{-\frac{1}{3}}x^{-\frac{1}{3}})\\ &= T_0\gamma^{\frac{1}{3}}D^{-\frac{1}{3}}x^{-\frac{1}{3}}\frac{\partial}{\partial y}(f'(\eta(x,y))) = T_0\gamma^{\frac{1}{3}}D^{-\frac{1}{3}}x^{-\frac{1}{3}}f''(\eta(x,y))\frac{\partial \eta(x,y)}{\partial y}\\ &= T_0\gamma^{\frac{2}{3}}D^{-\frac{2}{3}}x^{-\frac{2}{3}}f''(\eta(x,y)).\end{aligned}$$

Substitute these back to the PDE, we get (here, f' represents $f'(\eta)$ and etc.)

$$\begin{aligned}\gamma y T_x &= DT_{yy}\\ \gamma y\left(-\frac{1}{3}\right)T_0 f' y\gamma^{\frac{1}{3}}D^{-\frac{1}{3}}x^{-\frac{4}{3}} &= DT_0\gamma^{\frac{2}{3}}D^{\frac{2}{3}}x^{-\frac{2}{3}}f''\\ -\frac{1}{3}\left(\frac{y^2\gamma^{\frac{2}{3}}}{D^{-\frac{2}{3}}x^{\frac{2}{3}}}\right)f' &= f''\\ -\frac{\eta^2}{3}f'(\eta) &= f''(\eta).\end{aligned} \tag{208}$$

Thus, we have obtained an ODE for f.
Remark: If your choice of $L(x)$ and $a(x)$ are correct, then the above must reduce to an ODE. If you can't cancel the physical constants such that you get just terms in η, you might have made a mistake in your choice of $L(x)$ and/or $a(x)$.

Step 4: Determine the corresponding boundary conditions for your ODE in f. The following shows the corresponding conversion of the given auxiliary boundary conditions to the corresponding boundary conditions for the ODE in f:

- $T(x,0) = T_0 \to T(x,0) = a(x)f(0) = T_0 f(0) = T_0$. This implies that $f(0) = 1$.
- $T(0,y) = T_0 f\left(\frac{y}{L(0)}\right) = T_0 f\left(\frac{y}{0}\right) = 0$. This implies that $\lim_{\eta\to\infty} f(\eta) = 0$ (ie "$f(\infty) = 0$").
- $T(x,\infty) = T_0 f\left(\frac{\infty}{L(x)}\right) = 0$. This also implies that $\lim_{\eta\to\infty} f(\eta) = 0$ (ie "$f(\infty) = 0$").

Step 5: Solve the given ODE: Let $g(\eta) = f'(\eta)$. This implies that we have

$$g'(\eta) = -\frac{\eta^2}{3}g(\eta). \tag{209}$$

One can solve this by separation of variables as follows:

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}\eta} &= -\frac{\eta^2}{3}g \\ \frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\eta} &= -\frac{\eta^2}{3} \\ \int \frac{1}{g}\mathrm{d}g &= \int -\frac{\eta^2}{3}\mathrm{d}\eta \\ \ln|g| &= -\frac{\eta^3}{6} + C \\ g(\eta) &= Ae^{-\frac{\eta^3}{6}}.\end{aligned} \tag{210}$$

Here, $A = \pm e^C$ is an arbitrary constant. Now, substitute back $f'(\eta) = g(\eta)$ to obtain

$$f'(\eta) = Ae^{-\frac{\eta^3}{6}}. \tag{211}$$

Recall that $f(0) = 1$, so we can use the Fundamental theorem of Calculus by integrating the above ODE with respect to η starting from 0. This yields

$$\begin{aligned}\int_0^\eta f'(\xi)\mathrm{d}\xi &= \int_0^\eta Ae^{-\frac{\xi^3}{6}}\mathrm{d}\xi \\ f(\eta) - f(0) &= A\int_0^\eta e^{-\frac{\xi^3}{6}}\mathrm{d}\xi \\ f(\eta) &= 1 + A\int_0^\eta e^{-\frac{\xi^3}{6}}\mathrm{d}\xi.\end{aligned} \tag{212}$$

Using the other boundary condition, ie $f(\infty) = 0$, we have

$$\begin{aligned}0 &= 1 + A\int_0^\infty e^{-\frac{\xi^3}{6}}\mathrm{d}\xi \\ A &= \frac{-1}{\int_0^\infty e^{-\frac{\xi^3}{6}}\mathrm{d}\xi}.\end{aligned} \tag{213}$$

This implies that we have

$$\begin{aligned}f(\eta) &= 1 - \frac{\int_0^\eta e^{-\frac{\xi^3}{6}}\mathrm{d}\xi}{\int_0^\infty e^{-\frac{\xi^3}{6}}\mathrm{d}\xi} \\ f\left(\frac{y}{\left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}}\right) &= 1 - \frac{\int_0^{\left(\frac{y}{\left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}}\right)} e^{-\frac{\xi^3}{6}}\mathrm{d}\xi}{\int_0^\infty e^{-\frac{\xi^3}{6}}\mathrm{d}\xi}.\end{aligned} \tag{214}$$

The final similarity solution for the PDE is thus given by

$$\begin{aligned}T(x,y) &= T_0 f(\eta(x,y)) \\ &= T_0\left(1 - \frac{\int_0^{\left(\frac{y}{\left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}}\right)} e^{-\frac{\xi^3}{6}}\mathrm{d}\xi}{\int_0^\infty e^{-\frac{\xi^3}{6}}\mathrm{d}\xi}\right).\end{aligned} \tag{215}$$

[a] It really is. To see that it is the case, try setting $f(0) = 2$ and see if the final solution derived for the PDE changes.

Remark: In Step 2, if the condition given is in the form of a conservation of mass of some sort, say

$$\int T(x,y)\mathrm{d}x = \kappa \tag{216}$$

for all y, we can obtain $a(x)$ via dimensional analysis. Note that an integral represents the sum of product of T and x, so

$$\begin{aligned} [T][x] &= [\kappa] \\ [a] = [T] &= \frac{[\kappa]}{[x]} \end{aligned} \tag{217}$$

This suggest using $a(x)$ of the form

$$a(x) = \frac{\kappa}{x}. \tag{218}$$

For this case, the actual scaling constant of $a(x)$ will be captured by the value of $f(0)$ or by employing relevant initial conditions etc.

If the final unit that you've obtained is dependent on both x and y but you need $a(x)$, recall that you have $[y] = f([x])$ from (205).

Fundamental Solutions to the $1D-$Diffusion/Heat Equation.
Consider the $1D-$diffusion equation on (an **unbounded spatial domain**) $\mathbb{R}$ as described mathematically below.

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{219}$$

We say that $u(x,t) = S(x,t)$ is the fundamental solution[12] to (219) if

- $S_t(x,t) = DS_{xx}(x,t)$ for all $x \in \mathbb{R}, t > 0$ (inequality is strict here), and
- $\int_{-\infty}^{\infty} S(x,0)\,\mathrm{d}x = 1$. (This follows from conservation of mass, as $\int_{-\infty}^{\infty} S(x,t) = 1$ for all $t > 0$.)

It has been derived in lecture that $S(x,t)$ takes the form

$$S(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \tag{220}$$

for $t > 0$. Now, note that $S(x,t)$ might not solve (219) for any given initial data $u(x,0) = \phi(x)$. Instead, one can derive the following solution to (219):

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y. \tag{221}$$

This will be helpful in trying to solve variants of the $1D-$diffusion equation in $\mathbb{R} \times [0,\infty)$.

[12] As mentioned in the lectures, there is a slightly more rigorous way to define this with the notion of a dirac delta distribution. For this class, we shall not do so, and work with the definition below.

Example 48. Solve the following PDEs for $u(0,t)$, for $t>0$:

(i)

$$\begin{cases} u_t(x,t) = 2022u_{xx}(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = x & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{222}$$

(ii)

$$\begin{cases} u_t(x,t) = 3u_{xx}(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = x^2 & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{223}$$

Suggested Solution:

(i) Using the source solution formula in (221) with $D=2022$, we have for any $t>0$,

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{8088\pi t}} e^{-\frac{|x-y|^2}{8088t}} y\,\mathrm{d}y \\ u(0,t) &= \frac{1}{\sqrt{8088\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{8088t}} y\,\mathrm{d}y \\ &= -\frac{4044t}{\sqrt{8088\pi t}} \int_{-\infty}^{\infty} \left(\frac{-y}{4044t}\right) e^{-\frac{y^2}{8088t}}\,\mathrm{d}y \\ &= -\frac{4044t}{\sqrt{8088\pi t}} \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}y} e^{-\frac{y^2}{8088t}}\,\mathrm{d}y \quad \text{by reverse chain rule} \\ &= -\frac{4044t}{\sqrt{8088\pi t}} e^{-\frac{y^2}{8088t}} \Big|_{y=-\infty}^{y=\infty} \quad \text{by Fundamental Theorem of Calculus} \\ &= 0. \end{aligned} \tag{224}$$

(i) Using the source solution formula in (221) with $D=3$, we have for any $t>0$,

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{12\pi t}} e^{-\frac{|x-y|^2}{12t}} y^2\,\mathrm{d}y \\ u(0,t) &= \frac{1}{\sqrt{12\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{12t}} y^2\,\mathrm{d}y \\ &= \frac{-6t}{\sqrt{12\pi t}} \int_{-\infty}^{\infty} \left(\frac{-2y}{12t}\right) e^{-\frac{y^2}{12t}} y\,\mathrm{d}y \\ &= \frac{-6t}{\sqrt{12\pi t}} \int_{-\infty}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}y} e^{-\frac{y^2}{12t}}\right) y\,\mathrm{d}y \quad \text{by reverse chain rule} \\ &= \frac{-6t}{\sqrt{12\pi t}} \left(e^{-\frac{y^2}{12t}} y \Big|_{y=-\infty}^{y=\infty} - \int_{-\infty}^{\infty} e^{-\frac{y^2}{12t}}\,\mathrm{d}y \right) \quad \text{by integration by parts} \\ &= \frac{6t}{\sqrt{12\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{12t}}\,\mathrm{d}y \quad \text{boundary terms vanish} \\ &= 6t \int_{-\infty}^{\infty} \frac{1}{\sqrt{12\pi t}} e^{-\frac{y^2}{12t}}\,\mathrm{d}y \\ &= 6t. \end{aligned} \tag{225}$$

Note that the boundary term vanishes since the exponential term $e^{-\frac{y^2}{12t}}$ goes to 0 as $y\to\pm\infty$ faster than any polynomial, and thus $e^{-\frac{y^2}{12t}}y \to 0$ as $y\to\pm\infty$. The last equality comes from the fact that as a fundamental solution, we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{|x-y|^2}{4Dt}}\,\mathrm{d}y = 1 \tag{226}$$

for all $x\in\mathbb{R}, t>0$. We shall use this formula for $x=0$ and $D=3$ to obtain $\int_{-\infty}^{\infty} \frac{1}{\sqrt{12\pi t}} e^{-\frac{y^2}{12t}} = 1$.

Remark: (Optional for those without a background in probability theory, ie Math 170E.)

As you might have noticed,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{8088\pi t}} e^{-\frac{|x-y|^2}{8088t}} y \ \mathrm{d}y \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(\sqrt{4044t})^2}} e^{-\frac{|x-y|^2}{2(\sqrt{4044t})^2}} y \ \mathrm{d}y \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|y-\mu|^2}{2\sigma^2}} y \ \mathrm{d}y \\
&= \mathbb{E}(X),\ X \sim N(\mu, \sigma^2) = N(x, (\sqrt{4044t})^2) = N(x, 4044t).
\end{aligned} \tag{227}$$

Thus, the above expression in (i) (without substituting $x = 0$) refers to evaluating the mean of a normal distribution with mean x and variance $4044t$. For our case, since $X \sim N(0, 4044t)$, it is indeed expected that $\mathbb{E}(X) = 0$, as an elementary fact from Math 170E. Similarly, we can see that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{12\pi t}} e^{-\frac{|x-y|^2}{12t}} y^2 \ \mathrm{d}y \\
&= \mathbb{E}(X^2),\ X \sim N(\mu, \sigma^2) = N(x, 6t). \\
&= \mathbb{V}ar(X) - \mathbb{E}(X)^2.
\end{aligned} \tag{228}$$

The last equality follows from a standard property for expectation and variance, mainly,

$$\mathbb{V}ar(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2. \tag{229}$$

For our case, we have $X \sim N(0, 6t)$, thus $\mathbb{E}(X) = 0$. The integral evaluated in (ii) corresponds to asking for the variance of X with mean 0. This is nothing but $6t$.

Example 49. Derive a formula for source solution for the $1D-$diffusion equation (with $D > 0$, a constant) with an additional source term, given by

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) + t^2 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{230}$$

Suggested Solution: An observation to note is that if one shifts the t^2 term to the left, we have

$$\begin{aligned} u_t - t^2 &= Du_{xx} \\ \left(u - \frac{t^3}{3}\right)_t &= Du_{xx} \\ \left(u - \frac{t^3}{3}\right)_t &= D\left(u - \frac{t^3}{3}\right)_{xx}. \end{aligned} \tag{231}$$

The last equality comes from the fact that the partial derivatives in x are linear, and that $-\frac{t^3}{3}$ is independent of x and thus vanishes. This suggests the use of the following substitution:

$$v(x,t) = u(x,t) - \frac{t^3}{3}. \tag{232}$$

The new PDE in $v(x,t)$ can be obtained directly from above. The new initial condition can be evaluated using the subsitution as follows:

$$v(x,0) = u(x,0) - \frac{0^3}{3} = u(x,0) = \phi(x). \tag{233}$$

We thus now have

$$\begin{cases} v_t(x,t) = Dv_{xx}(x,t) & \text{in } \mathbb{R} \times (0,\infty), \\ v(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{234}$$

Now, we can apply the source formula (221) for the solution to the above $1D-$diffusion equation on $\mathbb{R}$ to obtain

$$v(x,t) = \int_{\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y. \tag{235}$$

Substituting v back for u, we have for $(x,t) \in \mathbb{R} \times (0,\infty)$,

$$u(x,t) = \frac{t^3}{3} + \int_{\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y. \tag{236}$$

Here, $S(x,t)$ for any $x \in \mathbb{R}$, $t > 0$, is given in (220).

7 Discussion 7.

All functions have sufficient smoothness as required, unless stated otherwise.

Comparing Transport, Diffusion and Wave Equations.

Recall that the relevant PDEs are given by

- Transport Equation (or just First Order PDEs) $u_t(x,t) + a(x,t)u_x(x,t) = b(x,t)$,
- Diffusion Equation $u_t(x,t) = Du_{xx}(x,t)$, and
- Wave Equation $u_{tt}(x,t) = c^2 u_{xx}(x,t)$.

Key phenomena:

- Transport equation corresponds to "transporting" material (with density described by u) along characteristic curves with speed $\frac{\mathrm{d}x}{\mathrm{d}t} = a(x,t)$.
- Diffusion equation smoothen out any initial data with infinite speed of propagation. This can be seen from the solution to the initial value problem, where

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4Dt}} \phi(y) \mathrm{d}y$$

  is smooth even if $\phi(y)$ is discontinuous![13]
- Diffusion equation obeys maximum (minimum) principle.
- Wave equation has solutions in which the speed of propagation is finite, at c. This can be seen from the d'Alembert's solution, given by

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) \mathrm{d}y.$$

  For instance, in Homework 4 Exercise 3 in which $\phi(x) = 1 - |x|$ if $|x| < 1$ and 0 if otherwise, and $\psi(x) = 0$, we have deduced using the spacetime diagram that the string at $x = 3$ first starts to move at $t = 2$ when $c = 1$, and $u(3,t) = 0$ is zero for any time $0 \le t < 2$.

The list above is not exhaustive, and draws knowledge from your exposure in these types of equations in lectures from the past weeks.

[13] From a mathematical analysis point of view, the smoothness of this function can be proven by literally taking limits $\frac{u(x+\Delta x) - u(x)}{\Delta x}$ as $\Delta x \to 0$.

Example 50. Let $u(x,t)$ model the density of some sort of particle, with $x \in \mathbb{R}$ and $t > 0$. For the scenarios given below, explain whether the information supports the density of the particle being modeled using

(i) A transport equation,

(ii) A diffusion equation, or

(iii) A wave equation.

Explain each of your answers briefly. In some cases, more than one type of equation may be appropriate.

(a) A sharp disturbance at $x = 0$ at time $t = 0$ is captured by two detectors placed at $x = -10$m and $x = 10$m **only** at time $t = 5$s.

Suggested Solution :

(a)(i) The transport equation is not appropriate for this case, as we should expect the disturbance to be detected only on one of the detector (along a characteristic curve).

(ii) The diffusion equation is not appropriate, as it smoothens out any initial data (and thus disturbances), and thus, the disturbance should not only be captured by detectors that are significantly far apart, and only at a specific time.

(iii) Yes of course, as this follows immediately from d'Alembert's formula! In fact, one can deduce that the value of u detected at each detector will be exactly half of the value that it started out with. Furthermore, we can also determine the speed of propagation, that is, $c = \frac{10\text{m}}{5\text{s}} = 2 \text{ m s}^{-1}$.

Solving Diffusion and Wave Equations on the Half-Line.

For this section of the supplement, we shall consider solving the diffusion and wave equations on the half line. As the homework questions relating to these concepts are theoretical in nature, so would the focus on this part of the supplement be. We will end off by throwing in a computational example as an exercise towards the end.

First, we consider the diffusion equation on the half-line with $u(x,0)=\phi(x)$. Mathematically, this is given by

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } [0,\infty)\times(0,\infty), \\ u(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } [0,\infty)\times\{t=0\}. \end{cases} \tag{237}$$

By odd extension, we can consider extending $u(x,t)$ and its initial data $\phi(x)$ for $x<0$. We then obtain the solution for $x\geq 0$ simply by restricting the final solution to $x \geq 0$. Mathematically, solving (237) is equivalent to solving

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = \phi_{\text{odd}}(x) & \text{on } \mathbb{R}\times\{t=0\}, \end{cases} \tag{238}$$

where

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x>0, \\ -\phi(-x) & \text{for } x<0, \\ 0 & \text{for } x=0. \end{cases} \tag{239}$$

Using our fundamental solution, we can then obtain solution to (238) by

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\mathrm{d}y \\ &= \int_{0}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\mathrm{d}y + \int_{-\infty}^{0} S(x-y,t)\phi_{\text{odd}}(y)\mathrm{d}y \\ &= \int_{0}^{\infty} S(x-y,t)\phi(y)\mathrm{d}y - \int_{-\infty}^{0} S(x-y,t)\phi(-y)\mathrm{d}y \\ &= \int_{0}^{\infty} S(x-y,t)\phi(y)\mathrm{d}y + \int_{\infty}^{0} S(x+y,t)\phi(y)\mathrm{d}y \text{ using substitution } y\mapsto -y \\ &= \int_{0}^{\infty} (S(x-y,t)-S(x+y,t))\phi(y)\mathrm{d}y. \end{aligned} \tag{240}$$

Now, we can just evaluate this function for any given $x \geq 0$.

Remark: For Neumann boundary conditions, consider an even extension; ie,

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{for } x>0, \\ \phi(-x) & \text{for } x<0, \\ 0 & \text{for } x=0. \end{cases} \tag{241}$$

Another Remark: A possible aid to understand even and odd extensions -

- For odd extensions, one can check that the new solution $u(x,t)$ is odd in x (ie if $u(x,t)$ satisfies the PDE, then $-u(-x,t)$ satisfies the PDE too), for any $t \geq 0$. Then, we recall that any odd function $f(x)$ must satisfy $f(0) = 0$.[14] This thus implies that $u(0,t) = 0$ for any $t \geq 0$.
  In particular, $u(x,0) = \phi(x)$ must be odd, and thus, this indicates that we might employ an odd extension (reflection) here.
- For even extensions, one can check that the new solution $u(x,t)$ is even in x (ie if $u(x,t)$ satisfies the PDE, then $-u(x,t)$ satisfies the PDE too), for any $t \geq 0$. Then, we recall that any even function $f(x)$ must satisfy $f'(0) = 0$.[15]This implies that $u_x(0,t) = 0$ is satisfied for all $t \geq 0$ (corresponding to the Neumann boundary condition at $x = 0$).
  In particular, $u(x,0) = \phi(x)$ must be even, and thus, this indicates that we might employ an even extension here.

Using a similar idea, we can solve the wave equation on the half line with values on the boundary $x = 0$ to be 0. Mathematically, this is given by

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) & \text{in } [0,\infty) \times (0,\infty), \\ u(0,t) = 0 & \text{on } \{x = 0\} \times [0,\infty), \\ u(x,0) = \phi(x) & \text{on } [0,\infty) \times \{t = 0\}, \\ u_t(x,0) = \psi(x) & \text{on } [0,\infty) \times \{t = 0\}. \end{cases} \tag{242}$$

Recognize that we are given the Dirichlet boundary condition ($u(0,t) = 0$ for all $t \geq 0$) on $x = 0$, we consider the odd extension **for both** ϕ and ψ. Thus, solving (242) is equivalent to solving

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) & \text{in } [0,\infty) \times (0,\infty), \\ u(x,0) = \phi_{\text{odd}}(x) & \text{on } [0,\infty) \times \{t = 0\}, \\ u_t(x,0) = \psi_{\text{odd}}(x) & \text{on } [0,\infty) \times \{t = 0\}, \end{cases} \tag{243}$$

in which d'Alembert's solution yields

$$u(x,t) = \frac{1}{2}\left(\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)\mathrm{d}y. \tag{244}$$

Using a similar argument as for the diffusion equation (by unpacking the "oddness" in the function), one should be able to obtain

$$u(x,t) = \begin{cases} \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\mathrm{d}y & \text{for } x > c|t| \\ \frac{1}{2}(\phi(ct+x) - \phi(ct-x)) + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)\mathrm{d}y & \text{for } 0 < x < c|t| \end{cases} \tag{245}$$

While unpacking, for instance for ϕ_{odd}, it might be helpful to think of cases in which the argument is positive and negative. This should yield the two separate cases as seen from above. (For more details, see Strauss's derivation/Marcus's derivation from 6.2.pdf.)

Intuitively, the solution for $x > c|t|$ does not have to rely on the method of odd extension/reflection about the boundary $x = 0$, so it follows from d'Alembert's formula. For $0 < x < c|t|$, this is reached by waves going to the left and hitting the boundary $x = 0$, and then reflected upwards to arrive at this domain. The reflection of wave about $x = 0$ introduces an additional negative sign (as seen in the negative sign in front of the term $\phi(ct - x)$, corresponding to waves going to the left at speed c; contribution from ψ is more subtle).
The exact details for dealing with Neumann boundary conditions at $x = 0$ will be the theme for Exercises 3 and 4 in Homework 6.

[14]Proof: If f is odd, then $f(x) = -f(-x)$ for all $x \geq 0$. In particular, we must have $f(0) = -f(-0) = -f(0)$, which implies $2f(0) = 0$ and thus $f(0) = 0$.

[15]Proof: If f is even, then $f(x) = f(-x)$ for all $x \geq 0$. Taking derivatives on both sides, we have $f'(x) = -f'(-x)$. In particular, we must have $f'(0) = -f'(-0) = -f'(0)$, which implies $2f'(0) = 0$ and thus $f'(0) = 0$.

Example 51. (Strauss 3.1.1.) Solve $u(x,t)$ on the half-line (ie $x>0, t>0$), for

$$\begin{cases} u_t(x,t) = ku_{xx}(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ u(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u(x,0) = e^{-x} & \text{on } (0,\infty)\times\{t=0\}. \end{cases} \tag{246}$$

Suggested Solution :

Using the formula as derived from (240), we have

$$\begin{aligned} u(x,t) &= \int_0^\infty (S(x-y,t) - S(x+y,t))\phi(y)\mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}\right) e^{-y}\mathrm{d}y. \end{aligned} \tag{247}$$

The first integral can be evaluated as follows:

$$\begin{aligned} &\int_0^\infty e^{-\frac{(x-y)^2}{4kt} - y}\mathrm{d}y \\ &= \int_0^\infty e^{-\frac{x^2 - 2xy + y^2 + 4kyt}{4kt}}\mathrm{d}y \\ &= e^{-\frac{x^2}{4kt}} \int_0^\infty e^{-\frac{y^2 - 2(x-2kt)y}{4kt}}\mathrm{d}y \\ &= e^{-\frac{x^2}{4kt}} \int_0^\infty e^{-\frac{(y-(x-2kt))^2 + (x-2kt)^2}{4kt}}\mathrm{d}y \\ &= e^{-\frac{x^2 - (x-2kt)^2}{4kt}} \int_0^\infty e^{-\frac{(y-(x-2kt))^2}{4kt}}\mathrm{d}y \\ &= e^{-\frac{(2kt)(2x-2kt)}{4kt}} \int_{-(x-2kt)}^\infty e^{-\frac{z^2}{4kt}}\mathrm{d}z \ \text{ using } z = y-(x-2kt) \\ &= e^{kt-x}\sqrt{4kt} \int_{-\frac{(x-2kt)}{\sqrt{4kt}}}^\infty e^{-w^2}\mathrm{d}w \ \text{ using } w = \frac{z}{\sqrt{4kt}} \\ &= \sqrt{\pi kt}e^{kt-x}\frac{2}{\sqrt{\pi}} \int_{-\frac{(x-2kt)}{\sqrt{4kt}}}^\infty e^{-w^2}\mathrm{d}w \\ &= \sqrt{\pi kt}e^{kt-x}\text{erfc}\left(-\frac{(x-2kt)}{\sqrt{4kt}}\right). \end{aligned} \tag{248}$$

Here, the error function (erf) and the complementary error function (erfc) is given by

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\mathrm{d}t, \quad \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\mathrm{d}t, \tag{249}$$

and notice that $\text{erf}(z) + \text{erfc}(z) = 1$ for any $z \in \mathbb{R}$. These are special functions in which cannot be written as an explicit formula of known functions. Similarly, one can show that for the second integral, it is given by

$$\begin{aligned} &\int_0^\infty e^{-\frac{(x-y)^2}{4kt} + y}\mathrm{d}y \\ &= \sqrt{\pi kt}e^{kt+x}\left(1 + \text{erf}\left(\frac{(x+2kt)}{\sqrt{4kt}}\right)\right). \end{aligned} \tag{250}$$

Thus, the solution is given by

$$u(x,t) = \frac{1}{2}e^{kt-x}\text{erfc}\left(-\frac{(x-2kt)}{\sqrt{4kt}}\right) + \frac{1}{2}e^{kt+x}\left(1 + \text{erf}\left(\frac{(x+2kt)}{\sqrt{4kt}}\right)\right). \tag{251}$$

Green's Function for ODEs.

Before we begin, let us start with some "distributional" calculus, ie calculus involving the Heaviside-step function and Dirac-delta "functions". These can be made rigorous with the help of distribution theory, which we will not cover in this class.[16]

The Dirac-delta function $\delta(x)$ is defined such that $\delta(x) = 0$ for all $x \neq 0$, "undefined/∞ for $x = 0$", and $\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$. This corresponds to a "spike function" at $x = 0$. Since $\delta(x)$ is defined and is 0 for $x \neq 0$, we deduce that for all $a < 0$ and $b > 0$, we have

$$\int_a^b \delta(x)\mathrm{d}x = 1$$

(can be understood by removing an integral of value 0).

The Heaviside step function[17] $H(x)$ is defined such that

$$H(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases} \tag{252}$$

Now, convince yourself that

$$\frac{\mathrm{d}}{\mathrm{d}x} H(x) = \delta(x), \tag{253}$$

or equivalently (with the use of the "Fundamental Theorem of Calculus"),[18]

$$H(x) = \int_{-\infty}^{x} \delta(s)\mathrm{d}s. \tag{254}$$

Once you have convinced yourself that (252) holds, then, consider the following. We define the ReLU function[19] as

$$\text{ReLU}(x) = \begin{cases} x & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases} \tag{255}$$

Then, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\text{ReLU}(x) = \frac{\mathrm{d}}{\mathrm{d}x} H(x) = \delta(x), \tag{256}$$

or

$$\text{ReLU}(x) = \int_{-\infty}^{x} H(s)\mathrm{d}s. \tag{257}$$

Thus, note that the use of this "ReLU(x)" function is not necessary for solving a first-order ODE with forcing term.

Here, $\delta(x)$, $H(x)$ and ReLU(x) "measures" some level of discontinuity at the point of interest, say $x = 0$;

- ReLU(x) is continuous at $x = 0$, and thus any integral on some small interval with size going to 0 containing 0 will give 0,
- $H(x)$ has a finite jump discontinuity, with an integral on some small interval with size going to 0 giving a value of 0, and
- $\delta(x)$ has an "infinite" jump discontinuity such that the integral on some small interval with size going to 0 yielding a value of 1.

This is a key piece of intuition that we will use to derive some "continuity" condition as we attempt to solve inhomogoneous ODEs (and PDEs).

[16] If you are interested, one can take Math 251A for a good coverage on a theory of distribution, or Math 266B for a sufficient coverage for applied PDEs to be used to solve for "discontinuous" solutions. An amount equivalent to those covered in 266B is also covered in Strauss's Book, in Chapter 12.

[17] There are many conventions for this at $x = 0$; honestly, the value at 0 usually does not matter.

[18] When $x < 0$, the integral does not contain the origin, and thus gives a value of 0. However, when $x > 0$, the integral contains the origin now, and by the aforementioned property of the integral of a δ-function, we have that this is 1.

[19] The name of this function is not important, but I would just like to point your attention to how this is named. You might find this familiar if you took a class in Machine Learning, say Math 156! The intuition $\frac{\mathrm{d}}{\mathrm{d}x}\text{ReLU}(x) = H(x)$ is commonly used in signal processing.

Example 52. Find the general solution to the ODE for $t > 0$ below:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = ty(t) + f(t), \quad y(0) = 1. \tag{258}$$

Suggested Solution:

Step 1: Split the problem up so that the ODE to be solved has homogeneous boundary conditions[a].
By linearity, we can split the problem of solving the ODE in (258) into 2, namely,

$$\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} = ty_1(t), \quad y_1(0) = 1, \tag{259}$$

and

$$\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} = ty_2(t) + f(t), \quad y_2(0) = 0. \tag{260}$$

and see that $y(t) = y_1(t) + y_2(t)$ thus solves (258).
(259) can be solved by separation of variables. This yields

$$y_1(t) = Ae^{\frac{t^2}{2}}. \tag{261}$$

Use the initial conditions $y_1(0) = 1$ to obtain

$$y_1(t) = e^{\frac{t^2}{2}}. \tag{262}$$

Step 2: Construct ODE with forcing term and homogeneous boundary conditions for a slice of $f(s_i)$ with the help of delta functions.
Now, see that (260) is an easier problem to solve since we have the homogeneous boundary conditions. We shall attempt to solve this with the help of a delta function. Consider a modified problem for a fixed $s_i > 0$, with

$$\frac{\mathrm{d}y_2(t;s_i)}{\mathrm{d}t} = ty_2(t;s_i) + \Delta s f(s_i)\delta(t - s_i), \quad y_2(0;s_i) = 0. \tag{263}$$

Intuitively, we are decomposing the function $f(t)$ by delta-spikes, ie $f(t) = \lim_{\Delta s \to 0} f(s_i)\delta(t - s_i)\Delta s$. This is explained again in more details in Step 4.
Furthermore, $;s_i$ indicates that s_i is a parameter that the solution is dependent on. It also serves as a reminder that the ODE that we are solving is not the original ODE for $y_2(t)$. For $t \neq s_i$, $\delta(t - s_i) = 0$. Thus, we are asked to solve

$$\frac{\mathrm{d}y_2(t;s_i)}{\mathrm{d}t} = ty_2(t;s_i), \ \text{for } t \neq s_i, \quad y_2(0;s_i) = 0. \tag{264}$$

For clarity, we shall denote the solution for $y_2(t;s_i)$ for $0 \le t < s_i$ by $y_2^L(t;s_i)$, and the solution for $y_2(t;s_i)$ for $t > s_i$ by $y_2^R(t;s_i)$.
Step 3: Solve this ODE with the help of "continuity" conditions.
For $0 \le t < s_i$, we see that the solution is given by (we have solved the ODE without the boundary condition in as in (261))

$$y_2^L(t;s_i) = Be^{\frac{t^2}{2}}. \tag{265}$$

Note that $0 \le t < s_i$, so we can freely apply the boundary conditions at $t = 0$ (ie $y_2(0;s_i) = 0$). This should yield $B = 0$.

For $t > s_i$, we also have

$$y_2^R(t;s_i) = Ce^{\frac{t^2}{2}}. \tag{266}$$

This time round, we can't use the boundary conditions given to us, since y_2^R is solved for $t > s_i$, which is away from 0 (does not include 0). Nonetheless, we can use the "jump conditions", similar to those derived in class in a heuristic manner. This comes from (263) (since come to think of it, we have not used any information about the δ function except that it is 0 if the argument $t - s_i$ is non-zero). This is copied below:

$$\frac{\mathrm{d}y_2(t;s_i)}{\mathrm{d}t} = ty_2(t;s_i) + \Delta s f(s_i)\delta(t-s_i), \quad y_2(0;s_i) = 0 \tag{267}$$

- The δ-type discontinuity on the right must be matched with either the $\frac{\mathrm{d}y_2(t;s_i)}{\mathrm{d}t}$ term or the $y_2(t;s_i)$ term.
- If the δ-type discontinuity is placed on y_2, this implies that the type of discontinuity $\frac{\mathrm{d}y_2(t;s_i)}{\mathrm{d}t}$ is $\delta'(t-s_i)$ type, which is "super" singular, and more singular than $\delta(t-s_i)$ that we began with. This can't be matched with any other terms in (267). We then deduce that the δ-type discontinuity cannot fall on y_2 and must fall on $\frac{\mathrm{d}y_2(t;s_i)}{\mathrm{d}t}$.
- If $\frac{\mathrm{d}y_2(t;s_i)}{\mathrm{d}t} \sim \delta(t-s_i)$, using our "distributional" calculus rule, we have $y_2(t;s_i) \sim H(t-s_i)$, ie, a jump discontinuity! To determine the value of this jump, we go back to (267) and integrate for t going from $s_i - \varepsilon$ to $s_i + \varepsilon$. We thus have

$$\begin{aligned}\int_{s_i-\varepsilon}^{s_i+\varepsilon} \frac{\mathrm{d}y_2(t;s_i)}{\mathrm{d}t}\mathrm{d}t &= t\int_{s_i-\varepsilon}^{s_i+\varepsilon} y_2(t;s_i)\mathrm{d}t + \int_{s_i-\varepsilon}^{s_i+\varepsilon} \Delta s f(s_i)\delta(t-s_i)\mathrm{d}t \\ [y_2(t;s_i)]_{s_i^-}^{s_i^+} &= 0 + \Delta s f(s_i)\end{aligned} \tag{268}$$

  by sending $\varepsilon \to 0$, using "Fundamental Theorem of Calculus" in the first term, by the integral of H-type discontinuity about its discontinuous point of y_2 as $\varepsilon \to 0$ in the second term (see the remarks above before this example), and noting that Δs and $f(s_i)$ are constants, plus $\int_{s_i-\varepsilon}^{s_i+\varepsilon} \delta(t-s_i)\mathrm{d}t = 1$ for any $\varepsilon > 0$.

Summarizing what we have obtained,

$$y_2(t;s_i) = \begin{cases} y_2^L(t;s_i) = 0 & \text{for } 0 \le t < s_i \\ y_2^R(t;s_i) = Ce^{\frac{t^2}{2}} & \text{for } t > s_i \end{cases} \tag{269}$$

Now, we can apply our condition in (268) and solve for the corresponding value of C. This yields

$$\begin{aligned} Ce^{\frac{s_i^2}{2}} &= (\Delta s)f(s_i) \\ C &= (\Delta s)e^{-\frac{s_i^2}{2}}f(s_i). \end{aligned} \tag{270}$$

Explicitly, we have that the solution is given by

$$y_2(t;s_i) = \Delta s \begin{cases} 0 & \text{for } 0 \le t < s_i \\ f(s_i)e^{-\frac{s_i^2}{2}+\frac{t^2}{2}} & \text{for } t > s_i \end{cases} \tag{271}$$

and we define

$$\tilde{y_2}(t;s) := \begin{cases} 0 & \text{for } 0 \le t < s_i \\ f(s_i)e^{-\frac{s_i^2}{2}+\frac{t^2}{2}} & \text{for } t > s_i \end{cases} \tag{272}$$

so that $y_2(t;s_i) = (\Delta s)\tilde{y_2}(t;s_i)$.

Step 4: Construct the full solution to the ODE with the full $f(s)$.

From (271), it seems that for a "spike-like" forcing term given by $f(s_i)\Delta(s)\delta(t-s_i)$, the solution is only affected by it for $t > s_i$. Thus, for any given t, we only have to sum up the s_i's from above till we reach t, since any s_i before t will not affect the solution. Thus, we have

$$y_2(t) = \sum_{s_i<t} \tilde{y_2}(t;s_i)\Delta s. \tag{273}$$

In the limit as $\Delta s \to 0$, we have

$$y_2(t) = \int_0^t \tilde{y_2}(t;s)\mathrm{d}s. \tag{274}$$

Remark: Another way to think of it is that we retrieve the second ODE if we sum over these s_i, and sending $\Delta s \to 0$. In particular, from (263), summing over the i's, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \tilde{y_2}(t;s_i)\Delta s = t\sum_i \tilde{y_2}(t;s_i)\Delta s + \sum_i f(s_i)\delta(t-s_i)\Delta s \tag{275}$$

so under the limit as Δs goes to zero, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \tilde{y_2}(t,s)\mathrm{d}s &= t\int_0^\infty \tilde{y_2}(t;s_i)\mathrm{d}s + \int_0^\infty f(s)\delta(t-s)\mathrm{d}s,\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \tilde{y_2}(t,s)\mathrm{d}s &= t\int_0^\infty \tilde{y_2}(t;s_i)\mathrm{d}s + f(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}y_2 &= ty_2 + f(t),\end{aligned} \tag{276}$$

with $y_2 = \int_0^\infty \tilde{y_2}(t,s)\mathrm{d}s$. Then, use the fact that $\tilde{y_2} = 0$ for $s > t$ to reduce the integral to $y_2(t) = \int_0^t \tilde{y_2}(t,s)\mathrm{d}s$.
Thus, we have the solution to (281), the original ODE, to be

$$\begin{aligned} y(t) &= y_1(t) + y_2(t) = y_1(t) + \int_0^t \tilde{y_2}(t;s)\mathrm{d}s\\ &= e^{\frac{t^2}{2}} + \int_0^t f(s)e^{-\frac{s^2}{2}+\frac{t^2}{2}}\mathrm{d}s\\ &= e^{\frac{t^2}{2}}\left(1 + \int_0^t f(s)e^{-\frac{s^2}{2}}\mathrm{d}s\right)\end{aligned} \tag{277}$$

One can check that if we solve this equation (258) directly by integration factor, we should retrieve the above solution.

[a] That is, corresponding boundary conditions are the zero conditions.

Remark: If you are not comfortable with the approximation made in Step 4, here is an alternative way to think about it. If we instead looked at (263) without the Δs term, mainly

$$\frac{\mathrm{d}y_2(t;s)}{\mathrm{d}t} = ty_2(t;s) + f(s)\delta(t-s), \quad y_2(0;s) = 0, \tag{278}$$

there is no need to factor out Δs in (271). I have also dropped out the indices i since we are not doing "Riemann-sum-like" approximations now.

From the form in (278), we can integrate with respect to s from 0 to infinity (since $t > 0$)[20] to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty y_2(t;s)\mathrm{d}s &= t\int_0^\infty y_2(t;s)\mathrm{d}s + \int_0^\infty f(s)\delta(t-s)\mathrm{d}s\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty y_2(t;s)\mathrm{d}s &= t\int_0^\infty y_2(t;s)\mathrm{d}s + f(t).\end{aligned} \tag{279}$$

Thus, we see that $y_2(t) = \int_0^\infty y_2(t;s)\mathrm{d}s$ solves the ODE for y_2 in (260) directly! Furthermore, we can also check that $y_2(0) = \int_0^\infty y_2(0;s)\mathrm{d}s = 0$ since $y_2(0;s) = 0$ is the boundary condition that we imposed when we solve the ODE for $y_2(t;s)$.

Then, from (271) (without the Δs term since we did not include it in (279)), we see that $y_2(t;s) = 0$ if $t < s$ ie $s > t$. This cuts off the integral domain from $0 < t < \infty$ to $0 < t < s$. Hence, we have

$$y_2(t) = \int_0^t y_2(t;s)\mathrm{d}s = \int_0^\infty y_2(t;s)\mathrm{d}s, \tag{280}$$

thus retrieving back (274).

[20] At this point, since we started out being insufficiently rigorous with our "Fundamental Theorem of Calculus" and "Riemann-Sums", I don't think we have to "justify" an interchange between the time derivative and the integral!

I started out with this example, and quickly realized that it might be a little tedious. I have instead used the example above instead. Nonetheless, feel free to take a look at this if you would like to understand how to apply "distributional" calculus for higher order ODEs and see where "ReLU" is used.

Example 53. Find the general solution to the ODE for $t > 0$ below:

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = -y(t) + f(t), \quad y(0) = 0,\ y'(0) = 1. \tag{281}$$

Suggested Solution:

Step 1: Split the problem up so that the ODE to be solved has homogeneous boundary conditions[a].
By linearity, we can split the problem of solving the ODE in (281) into 2, namely,

$$\frac{\mathrm{d}^2 y_1(t)}{\mathrm{d}t^2} = -y_1(t), \quad y_1(0) = 0,\ y_1'(0) = 1. \tag{282}$$

and

$$\frac{\mathrm{d}^2 y_2(t)}{\mathrm{d}t^2} = -y_2(t) + f(t), \quad y_2(0) = 0,\ y_2'(0) = 0. \tag{283}$$

and see that $y(t) = y_1(t) + y_2(t)$ thus solves (281). Recall that the solution to (282) is given by (see Math 33B notes for a recap on this; you might see this ODE again for an upcoming concept a few weeks later)

$$y_1(t) = A\cos(t) + B\sin(t). \tag{284}$$

Use the initial conditions $y_1(0) = 0$ and $y_1'(0) = 1$ to obtain two equations in A and B. Solving them, one can obtain

$$y_1(t) = \sin(t). \tag{285}$$

Step 2: Construct ODE with forcing term and homogeneous boundary conditions for a slice of $f(s_i)$ with the help of delta functions.
Now, see that (283) is an easier problem to solve since we have the homogeneous boundary conditions. We shall attempt to solve this with the help of a delta function. Consider a modified problem for a fixed $s_i > 0$, with

$$\frac{\mathrm{d}^2 y_2(t; s_i)}{\mathrm{d}t^2} = -y_2(t; s_i) + \Delta s f(s_i)\delta(t - s_i), \quad y_2(0; s_i) = 0,\ y_2'(0; s_i) = 0. \tag{286}$$

Here, $; s_i$ indicates that s_i is a parameter that the solution is dependent on. It also serves as a reminder that the ODE that we are solving is not the original ODE for $y_2(t)$. For $t \neq s_i$, $\delta(t - s_i) = 0$. Thus, we are asked to solve

$$\frac{\mathrm{d}^2 y_2(t; s_i)}{\mathrm{d}t^2} = -y_2(t; s_i), \quad \text{for } t \neq s_i, \quad y_2(0; s_i) = 0,\ y_2'(0; s_i) = 0. \tag{287}$$

For clarity, we shall denote the solution for $y_2(t; s_i)$ for $0 \leq t < s_i$ by $y_2^L(t; s_i)$, and the solution for $y_2(t; s_i)$ for $t > s_i$ by $y_2^R(t; s_i)$.
Step 3: Solve this ODE with the help of "continuity" conditions.
For $0 \leq t < s_i$, we see that the solution is given by

$$y_2^L(t; s_i) = C\cos(t) + D\sin(t). \tag{288}$$

Note that $0 \leq t < s_i$, so we can freely apply the boundary conditions at $t = 0$ (ie $y_2(0; s_i) = y_2'(0; s_i) = 0$). This should yield $C = D = 0$.

For $t > s_i$, we also have

$$y_2^R(t; s_i) = E\cos(t) + F\sin(t). \tag{289}$$

This time round, we can't use the boundary conditions given to us, since y_2^R is solved for $t > s_i$, which is away from 0 (does not include 0). Nonetheless, we can use the "jump conditions", similar to those derived in class in a heuristic manner. This comes from (286) (since come to think of it, we have not used any information about the δ function except that it is 0 if the argument $t - s_i$ is non-zero). This is copied below:

$$\frac{\mathrm{d}^2 y_2(t;s_i)}{\mathrm{d}t^2} = -y_2(t;s_i) + \Delta s f(s_i)\delta(t-s_i), \quad y_2(0;s_i)=0,\ y_2'(0;s_i)=0. \tag{290}$$

- The δ-type discontinuity on the right must be matched with either the $\frac{\mathrm{d}^2 y_2(t;s_i)}{\mathrm{d}t^2}$ term or the $y_2(t;s_i)$ term.
- If the δ-type discontinuity is placed on y_2, this implies that the type of discontinuity $\frac{\mathrm{d}^2 y_2(t;s_i)}{\mathrm{d}t^2}$ is $\delta''(t-s_i)$ type, which is "super" singular, and more singular than $\delta(t-s_i)$ that we began with. This can't be matched with any other terms in (290). We then deduce that the δ-type discontinuity cannot fall on y_2 and must fall on $\frac{\mathrm{d}^2 y_2(t;s_i)}{\mathrm{d}t^2}$.
- If $\frac{\mathrm{d}^2 y_2(t;s_i)}{\mathrm{d}t^2} \sim \delta(t-s_i)$, using our "distributional" calculus rule, we have $\frac{\mathrm{d}y_2(t;s_i)}{\mathrm{d}t} \sim H(t-s_i)$ and $y_2(t;s_i) \sim \mathrm{ReLU}(t-s_i)$ (or since ReLU is basically continuous at $t=s_i$, then $y_2(t;s_i)$ is continuous at $t=s_i$).
- We thus obtain our first condition:

$$\begin{aligned} [y_2(t;s_i)]_{t=s_i^-}^{t=s_i^+} &:= y_2(s_i^+;s_i) - y_2(s_i^-;s_i) = \lim_{s\to s_i^+} y_2(s;s_i) - \lim_{s\to s_i^-} y_2(s;s_i) = 0 \\ \lim_{s\to s_i^+} y_2(s;s_i) &= \lim_{s\to s_i^-} y_2(s;s_i). \end{aligned} \tag{291}$$

- Next, we also know that $\frac{\mathrm{d}y_2(t;s_i)}{\mathrm{d}t}$ has a $H(t-s_i)$ type discontinuity - ie, a jump discontinuity! To determine the value of this jump, we go back to (290) and integrate for t going from $s_i-\varepsilon$ to $s_i+\varepsilon$. We thus have

$$\begin{aligned} \int_{s_i-\varepsilon}^{s_i+\varepsilon} \frac{\mathrm{d}^2 y_2(t;s_i)}{\mathrm{d}t^2}\mathrm{d}t &= -\int_{s_i-\varepsilon}^{s_i+\varepsilon} y_2(t;s_i)\mathrm{d}t + \int_{s_i-\varepsilon}^{s_i+\varepsilon} \Delta s f(s_i)\delta(t-s_i)\mathrm{d}t \\ [y_2'(t;s_i)]_{s_i^-}^{s_i^+} &= 0 + \Delta s f(s_i) \end{aligned} \tag{292}$$

by sending $\varepsilon \to 0$, using "Fundamental Theorem of Calculus" in the first term, continuity of y_2 in the second term, and noting that Δs and $f(s_i)$ are constants, plus $\int_{s_i-\varepsilon}^{s_i+\varepsilon} \delta(t-s_i)\mathrm{d}t = 1$ for any $\varepsilon > 0$. We thus obtain our second condition:

$$\begin{aligned} [y_2'(t;s_i)]_{s_i^-}^{s_i^+} &= \Delta s f(s_i) \\ \lim_{s\to s_i^+} y_2'(s;s_i) &= \lim_{s\to s_i^-} y_2'(s;s_i) + \Delta s f(s_i). \end{aligned} \tag{293}$$

Summarizing what we have obtained,

$$y_2(t;s_i) = \begin{cases} y_2^L(t;s_i) = 0 & \text{for } 0 \le t < s_i \\ y_2^R(t;s_i) = E\cos(t) + F\sin(t) & \text{for } t > s_i \end{cases} \tag{294}$$

Now, we can apply our conditions (291) and (293) to (294) and solve for the corresponding values of E and F!
(291) yields

$$E\cos(s_i) + F\sin(s_i) = 0. \tag{295}$$

(293) yields

$$-E\sin(s_i) + F\cos(s_i) = 0 + \Delta s f(s_i). \tag{296}$$

Solving these, we obtain

$$\begin{cases} E = -\Delta s f(s_i)\sin(s_i) \\ F = \Delta s f(s_i)\cos(s_i). \end{cases} \tag{297}$$

Explicitly, we have

$$y_2(t;s_i) = \Delta s \begin{cases} 0 & \text{for } 0 \le t < s_i \\ -f(s_i)\sin(s_i)\cos(t) + f(s_i)\cos(s_i)\sin(t) & \text{for } t > s_i \end{cases} \tag{298}$$

and we define

$$\tilde{y_2}(t;s) := \begin{cases} 0 & \text{for } 0 \le t < s_i \\ -f(s_i)\sin(s_i)\cos(t) + f(s_i)\cos(s_i)\sin(t) & \text{for } t > s_i \end{cases} \tag{299}$$

so that $y_2(t; s_i) = (\Delta s)\tilde{y_2}(t; s_i)$.

Step 4: Construct the full solution to the ODE with the full $f(s)$.

From (298), it seems that for a "spike-like" forcing term given by $f(s_i)\Delta(s)\delta(t - s_i)$, the solution is only affected by it for $t > s_i$. Thus, for any given t, we only have to sum up the s_i's till we reach t, since any s_i after t will not affect the solution. Thus, we have

$$y_2(t) = \sum_{s_i < t} \tilde{y_2}(t; s_i)\Delta s. \tag{300}$$

In the limit as $\Delta s \to 0$, we have

$$y_2(t) = \int_0^s \tilde{y_2}(t; s)\mathrm{d}s. \tag{301}$$

Thus, we have the solution to (281), the original ODE, to be

$$\begin{aligned} y(t) = y_1(t) + y_2(t) &= y_1(t) + \int_0^t \tilde{y_2}(t; s)\mathrm{d}s \\ &= \sin(t) + \int_0^t -f(s)\sin(s)\cos(t) + f(s)\cos(s)\sin(t)\mathrm{d}s \\ &= \sin(t)\left(1 + \int_0^t f(s)\cos(s)\mathrm{d}s\right) - \cos(t)\int_0^t f(s)\sin(s)\mathrm{d}s. \end{aligned} \tag{302}$$

[a] That is, corresponding boundary conditions are the zero conditions.

8 Discussion 8.

All functions have sufficient smoothness as required, unless stated otherwise.

Inhomogeneous Problems.

Instead of re-deriving the relevant formulas, we shall compile the list of formulas that one should take away with for this part of the content for 136.

Here, $\mathbb{R}$ = whole line, $\mathbb{R}^+$ = half line. The difference between homogeneous PDEs and inhomogeneous PDEs is with the addition of a source term $f(x,t)$ (independent of the unknown function $u(x,t)$).

Diffusion and Wave on $\mathbb{R}$.

	Diffusion on $\mathbb{R}$	Wave on $\mathbb{R}$
Homogeneous	i	iii
Inhomogeneous	ii	iv

With

(i) Homogeneous, Diffusion on $\mathbb{R}$. (Source Solution.)

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{303}$$

Solution: $\boxed{u(x,t) = \int_{\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y,}$ with $S(z,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{z^2}{4Dt}}$.

(ii) Inhomogeneous, Diffusion on $\mathbb{R}$. (Duhamel's Principle.)

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) + f(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{304}$$

Solution: $\boxed{u(x,t) = \int_{\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y + \int_0^t\int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,\mathrm{d}y\mathrm{d}s.}$

(iii) Homogeneous, Wave on $\mathbb{R}$. (d'Alembert's Solution.)

$$\begin{cases} u_{tt}(x,t) = c^2u_{xx}(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R}\times\{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{305}$$

Solution: $\boxed{u(x,t) = \frac{1}{2}\left(\phi(x-ct)+\phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\mathrm{d}y.}$

(iv) Inhomogeneous, Wave on $\mathbb{R}$. (Duhamel's Principle.)

$$\begin{cases} u_{tt}(x,t) = c^2u_{xx}(x,t) + f(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R}\times\{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{306}$$

Solution:

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\phi(x-ct)+\phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\mathrm{d}y + \frac{1}{2c}\int\int_{\Delta} f(y,s)\,\mathrm{d}y\mathrm{d}s. \\ &= \boxed{\frac{1}{2}\left(\phi(x-ct)+\phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\mathrm{d}y + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,\mathrm{d}y\mathrm{d}s.} \end{aligned}$$

Diffusion and Wave on $\mathbb{R}^+$.

	Diffusion on $\mathbb{R}^+$	Wave on $\mathbb{R}^+$
Homogeneous	v (Dirichlet), vi (Neumann)	ix (Dirichlet), x (Neumann)
Inhomogeneous	vii (Dirichelet), viii (Neumann)	xi (Dirichlet), xii (Neumann)

(v) Homogeneous, Diffusion on $\mathbb{R}^+$, Dirichlet BC[21].

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ u(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } (0,\infty)\times\{t=0\}. \end{cases} \tag{307}$$

Solution: Use odd extension, solve this on $\mathbb{R}$, and use the resulting solution on $x \in \mathbb{R}$ to $x \in \mathbb{R}^+$.

$$\boxed{u(x,t) = \int_{\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\,\mathrm{d}y}.$$

(vi) Homogeneous, Diffusion on $\mathbb{R}^+$, Neumann BC.

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ u_x(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } (0,\infty)\times\{t=0\}. \end{cases} \tag{308}$$

Solution: Use even extension, solve this on $\mathbb{R}$, and use the resulting solution on $x \in \mathbb{R}$ to $x \in \mathbb{R}^+$.

$$\boxed{u(x,t) = \int_{\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)\,\mathrm{d}y}.$$

(vii) Inhomogeneous, Diffusion on $\mathbb{R}^+$, Dirichlet BC. (Duhamel's Principle.)

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) + f(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ u(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } (0,\infty)\times\{t=0\}. \end{cases} \tag{309}$$

Solution: Use odd extension, solve this on $\mathbb{R}$, and use the resulting solution on $x \in \mathbb{R}$ to $x \in \mathbb{R}^+$.

$$\boxed{u(x,t) = \int_{\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f_{\text{odd in } x}(y,s)\,\mathrm{d}y\mathrm{d}s.}.$$

(viii) Inhomogeneous, Diffusion on $\mathbb{R}^+$, Neumann BC. (Duhamel's Principle.)

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) + f(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ u_x(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } (0,\infty)\times\{t=0\}. \end{cases} \tag{310}$$

Solution: Use even extension, solve this on $\mathbb{R}$, and use the resulting solution on $x \in \mathbb{R}$ to $x \in \mathbb{R}^+$.

$$\boxed{u(x,t) = \int_{\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f_{\text{even in } x}(y,s)\,\mathrm{d}y\mathrm{d}s.}.$$

(ix) Homogeneous, Wave on $\mathbb{R}^+$, Dirichlet BC.

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ u(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } (0,\infty)\times\{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } (0,\infty)\times\{t=0\}. \end{cases} \tag{311}$$

Solution: Use odd extension, solve this on $\mathbb{R}$, and use the resulting solution on $x\in\mathbb{R}$ to $x\in\mathbb{R}^+$.

$$u(x,t) = \frac{1}{2}\left(\phi_{\text{odd}}(x-ct) + \phi_{\text{odd}}(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)\mathrm{d}y.$$

(x) Homogeneous, Wave on $\mathbb{R}^+$, Neumann BC.

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ u_x(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } (0,\infty)\times\{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } (0,\infty)\times\{t=0\}. \end{cases} \tag{312}$$

Solution: Use even extension, solve this on $\mathbb{R}$, and use the resulting solution on $x\in\mathbb{R}$ to $x\in\mathbb{R}^+$.

$$u(x,t) = \frac{1}{2}\left(\phi_{\text{even}}(x-ct) + \phi_{\text{even}}(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(y)\mathrm{d}y.$$

(xi) Inhomogeneous, Wave on $\mathbb{R}^+$, Dirichlet BC. (Duhamel's Principle.)

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) + f(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ u(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } (0,\infty)\times\{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } (0,\infty)\times\{t=0\}. \end{cases} \tag{313}$$

Solution: Use odd extension, solve this on $\mathbb{R}$, and use the resulting solution on $x\in\mathbb{R}$ to $x\in\mathbb{R}^+$.

$$u(x,t) = \frac{1}{2}\left(\phi_{\text{odd}}(x-ct) + \phi_{\text{odd}}(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)\mathrm{d}y + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f_{\text{odd in } x}(y,s)\,\mathrm{d}y\mathrm{d}s.$$

(xii) Inhomogeneous, Wave on $\mathbb{R}^+$, Neumann BC. (Duhamel's Principle.)

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) + f(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ u_x(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } (0,\infty)\times\{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } (0,\infty)\times\{t=0\}. \end{cases} \tag{314}$$

Solution: Use even extension, solve this on $\mathbb{R}$, and use the resulting solution on $x\in\mathbb{R}$ to $x\in\mathbb{R}^+$.

$$u(x,t) = \frac{1}{2}\left(\phi_{\text{even}}(x-ct) + \phi_{\text{even}}(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(y)\mathrm{d}y + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f_{\text{even in } x}(y,s)\,\mathrm{d}y\mathrm{d}s.$$

Remarks:

- To remove the subscript $()_{\text{odd}}$ or $()_{\text{even}}$, one would have to unpack what is meant to be odd/even. Recall that we define $f_{\text{odd}}(x)$ and $f_{\text{even}}(x)$ as

$$f_{\text{odd}}(x) = \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \\ 0 & x = 0 \end{cases}, \text{ and } f_{\text{even}}(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}. \tag{315}$$

- Furthermore, the method that we use to derive all the above combinations follows a similar idea as in Discussion Supplement 7! You will be required to provide the relevant details in HW 7 Exercise 2.

- However, note that we are not encompassing the possibility of a Dirichlet/Neumann-like boundary conditions but with its value to be a function rather than just zero! The following example deals with such a case for the diffusion equation. The case for the wave equation is dealt with in HW 7 Exercise 3 (with a **slightly different method** - Hint for that exercise: Strauss Chapter 3 Page 78[22]).

- **Inhomogeneous PDEs and Durhamel's Principle**. Without loss of generality, we consider this for the diffusion equation (and without loss of generality, on $\mathbb{R}$). A similar argument holds for the wave equation. This is given by

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) + f(x,t) & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{316}$$

By linearity, this is equivalent to solving two separate PDEs, one with the forcing term and zero initial data, while the other without the forcing term but with the full initial data. These are given by

$$\begin{cases} (u_1)_t(x,t) = D(u_1)_{xx}(x,t) & \text{in } \mathbb{R} \times (0,\infty), \\ (u_1)(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{317}$$

and

$$\begin{cases} (u_2)_t(x,t) = D(u_2)_{xx}(x,t) + f(x,t) & \text{in } \mathbb{R} \times (0,\infty), \\ (u_2)(x,0) = 0 & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{318}$$

The solution to u_1 is given by the standard source formula for diffusion equation on the real line. However, to solve u_2, this is given by Durhamel's Principle (see derivation with Green's and delta functions in lectures). This implies that the solution is given by

$$\begin{aligned} u(x,t) &= u_1 + u_2 \\ &= \int_{\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{319}$$

- For the inhomogeneous wave equation, the solution includes a term

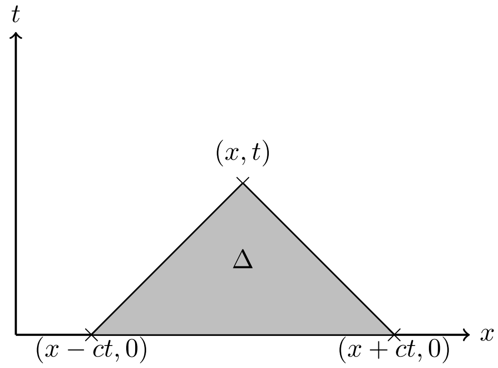
$$\frac{1}{2c}\int\int_{\Delta} f(y,s)\,\mathrm{d}y\mathrm{d}s = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,\mathrm{d}y\mathrm{d}s. \tag{320}$$

Here, Δ refers to the domain of dependence for a point (x,t).[23]

[22] Second Edition

[23] One has to take note of what this means in the case where we have $f_{\text{even in } x}$ or $f_{\text{odd in } x}$, as it is a folded rectangle. See lecture notes/Strauss's book for an example of this.

Diagram for Δ:



Example 54. Find the solution to the inhomogeneous diffusion equation with a boundary source $h(t)$, mainly,

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) + f(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ u(0,t) = h(t) & \text{on } \{x=0\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } (0,\infty)\times\{t=0\}. \end{cases} \tag{321}$$

You can leave your answers in $_{\text{even}}$ or $_{\text{odd}}$ without further simplification for this problem.

Suggested Solution:

Observe that

$$\begin{aligned} u_t &= Du_{xx} + f(x,t) \\ (u - h(t) + h(t))_t &= Du_{xx} + f(x,t) \\ (u - h(t))_t &= D(u-h(t))_{xx} + f(x,t) - h'(t) \end{aligned} \tag{322}$$

since $(h(t))_{xx} = 0$ as we are taking partial derivatives with respect to x on a function that only depends on t and not x. This suggests using a substitution

$$v(x,t) = u(x,t) - h(t). \tag{323}$$

From the above argument, we thus have

$$\begin{cases} v_t(x,t) = Dv_{xx}(x,t) + f(x,t) - h'(t) & \text{in } (0,\infty)\times(0,\infty), \\ v(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ v(x,0) = \phi(x) - h(0) & \text{on } (0,\infty)\times\{t=0\}, \end{cases} \tag{324}$$

since $v(x,0) = u(x,0) - h(0) = \phi(x) - h(0)$, and $v(0,t) = u(0,t) - h(t) = 0$ by (321). We can now apply case (vii) to obtain

$$\begin{aligned} u(x,t) &= \int_{\infty}^{\infty} S(x-y,t)(\phi - h(0))_{\text{odd}}(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)(f-h')_{\text{odd in } x}(y,s)\,\mathrm{d}y\mathrm{d}s. \\ &= \int_{\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f_{\text{odd in } x}(y,s)\,\mathrm{d}y\mathrm{d}s. \\ &+ \int_0^{\infty} S(x-y,t)h(0)\,\mathrm{d}y + \int_{-\infty}^{0} S(x-y,t)(-h(0))\,\mathrm{d}y - \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)h'(s)\,\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{325}$$

Since the question does not require further simplification (and it does not seem like much can be simplified, though it is possible), we are done!

Example 55. Find the solution to the inhomogeneous (forced) wave equation on $\mathbb{R}$ given by

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + f(x,t) & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = 0 & \text{on } \mathbb{R} \times \{t=0\}, \\ u_t(x,0) = 0 & \text{on } \mathbb{R} \times \{t=0\}, \end{cases} \tag{326}$$

with

$$f(x,t) = \begin{cases} 2 & x < 0 \text{ and } t > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{327}$$

Suggested Solution: This falls under case (iv). By the formula derived (or basically just d'Alembert's solution with Duhamel's Principle), with $c = 1$, we have since $\phi = \psi \equiv 0$,

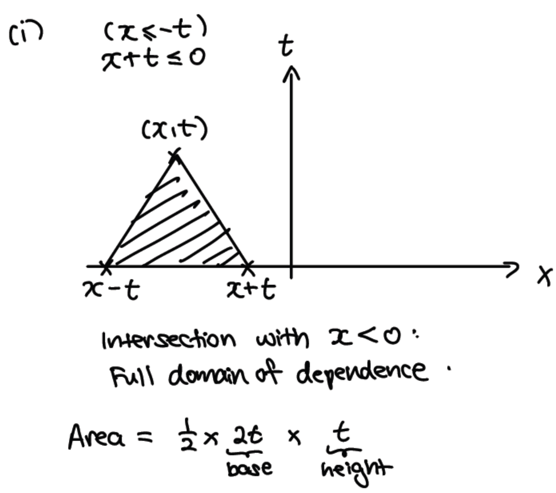
$$u(x,t) = \frac{1}{2} \iint_{\Delta} f(y,s)\,\mathrm{d}y\mathrm{d}s. \tag{328}$$

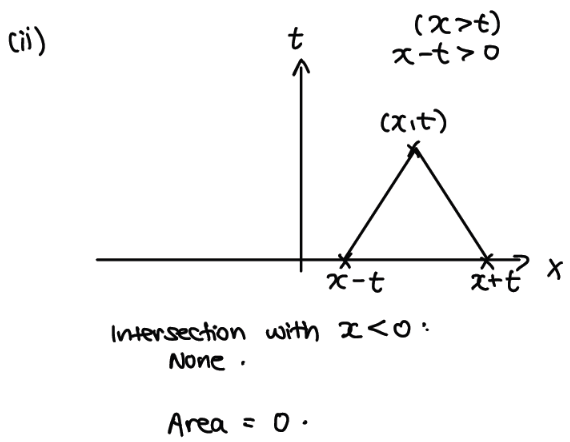
Instead of using the explicit formula for Δ, one can consider doing so by considering the physical intepretation of Δ as the domain of dependence, and its interaction with the forcing function f. See that

$$\begin{aligned} u(x,t) &= \frac{1}{2} \iint_{\Delta} f(y,s)\,\mathrm{d}y\mathrm{d}s. \\ &= \frac{1}{2} \iint_{\Delta \cap \{x<0\}} f(y,s)\,\mathrm{d}y\mathrm{d}s \quad \text{since } f = 0 \text{ on } x \geq 0 \\ &= \frac{1}{2} \iint_{\Delta \cap \{x<0\}} 2\,\mathrm{d}y\mathrm{d}s \quad \text{by definition of } f \\ &= \text{Area of } (\Delta \cap \{x < 0\}) \end{aligned} \tag{329}$$

Referring to the diagram below for the relevant geometric analysis, we have

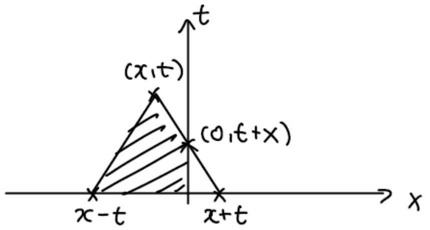
$$u(x,t) = \begin{cases} t^2 & x \leq -t \text{ (i)} \\ t^2 - \frac{1}{2}(x+t)^2 & -t < x \leq 0 \text{ (iii)} \\ \frac{1}{2}(t-x)^2 & 0 < x \leq t \text{ (iv)} \\ 0 & x > t \text{ (ii)} \end{cases} \tag{330}$$



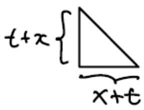


(iii) $\triangle$ intersect $\uparrow^{t}$, with (x,t) on left of $\uparrow^{t}$.

ie $-t < x \leq 0$



Area = Full triangle $-$ (minus)

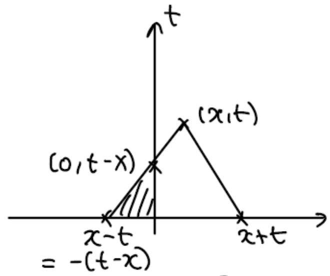


$= \frac{1}{2} \times 2t \times t - \frac{1}{2} \times (t+x) \times (x+t)$

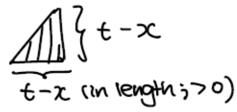
$= t^2 - \frac{1}{2}(x+t)^2.$

(iv) $\triangle$ intersect $\uparrow^{t}$, with (x,t) on right of $\uparrow^{t}$

ie $0 < x \leq t$.



Area =



$= \frac{1}{2}(t-x)^2.$

Primer to PDE by Separation of Variables (Boundary Value Problems)

Although the last exercise on HW 7 relates to this concept, we shall postpone the treatment of this till the next discussion section, as that will be the focus for HW 8/Discussion Supplement 9. Nonetheless, one should note that for this part of the course, you should recall how to solve standard ODEs from 33B.
Let us look at a couple of examples below (motivated by the possible ODEs that one would encounter in the upcoming lectures).

Example 56. Find all values of $\lambda > 0$ such that

$$X''(x) = -\lambda X(x) \text{ for } x \in (0,1), X(0) = X(1) = 0, \tag{331}$$

admits a non-zero solution on $x \in [0,1]$.

Suggested Solution:
Recall from 33B that the general solution is given by

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x). \tag{332}$$

Using $X(0) = 0$, we have $A = 0$. We then have

$$X(x) = B\sin(\sqrt{\lambda}x). \tag{333}$$

Next, using $X(1) = 0$, we have

$$B\sin(\sqrt{\lambda}) = 0. \tag{334}$$

Thus, it is possible to have $B \neq 0$ if $\sin(\sqrt{\lambda}) = 0$. This implies that we have[a]

$$\begin{aligned} \sqrt{\lambda} &= n\pi, n \in \mathbb{Z} \\ \lambda &= n^2\pi^2, n \in \mathbb{N} \setminus \{0\} \end{aligned} \tag{335}$$

Thus, the possible values of λ such that we have a non-zero solution on $[0,1]$ are

$$\boxed{\lambda = n^2\pi^2, n \in \mathbb{N} \setminus \{0\}}. \tag{336}$$

[a] $\mathbb{N}$ for me includes 0.

The purpose of this question, though serves as a recap, is to raise the following question. We know from 33B that the second-order ODE can be solved using the ansatz $e^{\mu x}$. Plugging this into (332), we have

$$\mu^2 e^{\mu x} = -\lambda e^{\mu x}. \tag{337}$$

Since $e^{\mu x} > 0$ for all real x, we then have $\mu^2 = -\lambda$, and we should obtain

$$\mu = \pm\sqrt{-\lambda} = \mathrm{i}\sqrt{\lambda}. \tag{338}$$

Thus, we should really be using

$$X(x) = Ae^{\mathrm{i}\sqrt{\lambda}x} + Be^{-\mathrm{i}\sqrt{\lambda}x}. \tag{339}$$

However, this might not be convenient for us, since both $X(0) = X(1)$ require use to solve a system of two equations with two unknowns (A and B). Instead, by Euler's formula $e^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)$, we have

$$\begin{aligned} X(x) &= A\cos(\sqrt{\lambda}x) + \mathrm{i}A\sin(\sqrt{\lambda}x) + B\cos(-\sqrt{\lambda}x) + \mathrm{i}B\sin(-\sqrt{\lambda}x) \\ &= A\cos(\sqrt{\lambda}x) + \mathrm{i}A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) - \mathrm{i}B\sin(\sqrt{\lambda}x) \\ &= (A+B)\cos(\sqrt{\lambda}x) + (\mathrm{i}A - \mathrm{i}B)\sin(\sqrt{\lambda}x) \\ &= A'\cos(\sqrt{\lambda}x) + B'\sin(\sqrt{\lambda}x) \end{aligned} \tag{340}$$

where $A' = A + B$, and $B' = \mathrm{i}A - \mathrm{i}B$ are the new arbitrary constants! This allows us to switch from the exponential form in (339) to (340)! Let us look at an example in which $\cosh$ and $\sinh$ are actually a little more useful in the example below.

Example 57. Find all values of $\lambda > 0$ such that

$$X''(x) = \lambda X(x) \text{ for } x \in (0,1), X(0) = X'(1) = 0, \tag{341}$$

admits a non-zero solution on $x \in [0,1]$.

Suggested Solution:
The general solution is given by

$$X(x) = A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x). \tag{342}$$

Using $X(0) = 0$, since $\sinh(0) = 0$ and $\cosh(0) = 1$, we have $A = 0$. We then have

$$X(x) = B\sinh(\sqrt{\lambda}x). \tag{343}$$

Next, we compute

$$X'(x) = B\sqrt{\lambda}\cosh(\sqrt{\lambda}x). \tag{344}$$

Next, using $X'(1) = 0$, we have

$$B\cosh(\sqrt{\lambda}) = 0. \tag{345}$$

Note that since $\cosh(x) \geq 1$ for all $x \in \mathbb{R}$, then $\cosh(\sqrt{\lambda}) \neq 0$ for all $\lambda \geq 0$. We can then drop this term on both sides to obtain $B = 0$.

Conclusion: No such λ exists.

Before we show that the general solution is indeed given by (342), let us recap some properties of sinh and cosh (the list of properties is not exhaustive):

- $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$.
- $\frac{\mathrm{d}}{\mathrm{d}x}\sinh(x) = \cosh(x)$, $\frac{\mathrm{d}}{\mathrm{d}x}\cosh(x) = \sinh(x)$ (Its just like derivatives for sin and cos, but without the negative sign when we differentiate cos).
- $\sinh(x) = 0 \implies x = 0$, while $\cosh(x) \geq 1$ for all $x \in \mathbb{R}$. See the graphs for $\sinh(x)$ and $\cosh(x)$.

Now, we shall derive why this is the case for (341). Use the ansatz $X(x) = e^{\mu x}$, plug into (341), drop $e^{\mu x}$, and obtain

$$\mu^2 = \lambda \implies \mu = \pm\sqrt{\lambda}. \tag{346}$$

The general solution is given by

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}. \tag{347}$$

This can be simplified as

$$\begin{aligned} X(x) &= Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\ &= \left(\frac{A+B}{2} + \frac{A-B}{2}\right)e^{\sqrt{\lambda}x} + \left(\frac{A+B}{2} - \frac{A-B}{2}\right)e^{-\sqrt{\lambda}x} \\ &= (A+B)\left(\frac{e^{\sqrt{\lambda}x} + e^{-\sqrt{\lambda}x}}{2}\right) + (A-B)\left(\frac{e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}}{2}\right) \\ &= A'\cosh(\sqrt{\lambda}x) + B'\sinh(\sqrt{\lambda}x), \end{aligned} \tag{348}$$

with new arbitrary constants $A' = A + B$, and $B' = A - B$.

9 Discussion 9.

All functions have sufficient smoothness as required, unless stated otherwise.

Remark: The way you should use this discussion is as such - there are various comments/footnotes on the right mathematical framework for the relevant concepts, though I would say that they are not essential. This is something that one can look into for the relevant technicalities/details, or if you are interested in the analysis aspect of these theories. Furthermore, such a perspective to this is unique to the level of details/perspective in this class, which might seem to be awkward if one is familiar with Fourier Analysis.

Boundary Value Problems.

For this discussion, the emphasis is on solving boundary value problems, i.e, considering a finite domain in space such that we impose certain values on the unknown function u and/or its derivative u_x on the boundary of the aforementioned space. In other words, we will be solving PDE problems (i.e say diffusion equation) that look like this

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } (a,b)\times(0,\infty), \\ u(a,t) = f(t) & \text{on } \{x=a\}\times[0,\infty), \\ u(b,t) = g(t) & \text{on } \{x=b\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } [a,b]\times\{t=0\}. \end{cases} \tag{349}$$

Before we try to solve some of these, here are some theoretical results (derived in class) that you should know.

Eigenvalue Problems for Self-Adjoint Differential Operator.

Recap: In linear algebra (33A or 115A), we say that we are solving for the eigenvalues and/or eigenvector of a real-valued $n \times n$ matrix if

$$A\mathbf{v} = \lambda\mathbf{v} \tag{350}$$

for some $\lambda \in \mathbb{R}$ and $\mathbf{v}$. We then call λ the eigenvalue, and $\mathbf{v} \neq \mathbf{0}$, the corresponding eigenvector for λ.[24] We can treat this concept analogously for a differential operator, by viewing $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and $\mathbf{v} = f$ as the (eigen)functions that can solve the ODE described by (350).

Inspired from Discussion Supplement 8 (and since I did not explicitly cover this in the previous discussion), let us look at an example below, similar to the last two examples covered in the previous discussion supplement, but worded using mathematical language that we are going to use for this discussion.

Example 58. Find all possible eigenvalues and eigenvectors for the differential operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ in the (vector) space of functions defined on $[0,1]$ satisfying the Dirichlet boundary conditions;[a] ie

$$\{\text{ functions } X \text{ on } [0,1] : X(0) = X(1) = 0\}.$$

In other words, find all possible values of $\lambda \in \mathbb{R}$ and the corresponding functions $X(x)$ such that

$$-X''(x) = \lambda X(x) \text{ for } x \in (0,1),\ X(0) = X(1) = 0, \tag{351}$$

admits a non-zero solution on its domain $x \in [0,1]$.

Suggested Solution:

$\lambda = 0$. This is equivalent to solving

$$-X''(x) = 0 \text{ for } x \in (0,1),\ X(0) = X(1) = 0. \tag{352}$$

Integrating with respect to x directly, we see that $X(x) = Ax + B$. Using the fact that $X(0) = X(1) = 0$, we can solve for A and B and obtain that $A = B = 0$.

[24] Technicality: An eigenvalue can be zero, but an eigenfunction cannot be the zero function since the associated eigenvalue for the zero function is just any real (complex) number!

$\underline{\lambda > 0.}$ This is equivalent to solving

$$X''(x) = -\lambda x \text{ for } x \in (0,1),\ X(0) = X(1) = 0. \tag{353}$$

Recall from 33B that the general solution is given by

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x). \tag{354}$$

Using $X(0) = 0$, we have $A = 0$. We then have

$$X(x) = B\sin(\sqrt{\lambda}x). \tag{355}$$

Next, using $X(1) = 0$, we have

$$B\sin(\sqrt{\lambda}) = 0. \tag{356}$$

Thus, it is possible to have $B \neq 0$ if $\sin(\sqrt{\lambda}) = 0$. This implies that we have[b]

$$\begin{aligned} \sqrt{\lambda} &= n\pi, n \in \mathbb{Z} \\ \lambda &= n^2\pi^2, n \in \mathbb{N} \setminus \{0\} \end{aligned} \tag{357}$$

Thus, for $\lambda > 0$, the corresponding eigenfunctions are $X_n(x) = \sin(n\pi x)$, with corresponding eigenvalues $\lambda_n = n^2\pi^2$ with $n \in \mathbb{N} \setminus \{0\}$.

$\underline{\lambda < 0.}$ This is equivalent to solving

$$X''(x) = \tilde{\lambda} x \text{ for } x \in (0,1),\ X(0) = X(1) = 0, \tag{358}$$

for $\tilde{\lambda} > 0$. From Discussion Supplement 8, the general solution is given by

$$X(x) = A\cosh(\sqrt{\tilde{\lambda}}x) + B\sinh(\sqrt{\tilde{\lambda}}x). \tag{359}$$

Using $X(0) = 0$, since $\sinh(0) = 0$ and $\cosh(0) = 1$, we have $A = 0$. We then have

$$X(x) = B\sinh(\sqrt{\tilde{\lambda}}x). \tag{360}$$

Next, we compute

$$X'(x) = B\sqrt{\tilde{\lambda}}\cosh(\sqrt{\tilde{\lambda}}x). \tag{361}$$

Next, using $X'(1) = 0$, we have

$$B\cosh(\sqrt{\tilde{\lambda}}) = 0. \tag{362}$$

Note that since $\cosh(x) \geq 1$ for all $x \in \mathbb{R}$, then $\cosh(\sqrt{\tilde{\lambda}}) \neq 0$ for all $\tilde{\lambda} > 0$. We can then drop this term on both sides to obtain $B = 0$. Thus, we necessarily obtain the zero solution. This implies that there are no eigenvalues (and thus corresponding eigenfunctions) in which $\lambda < 0$.

In summary, the eigenvalues with the corresponding eigenfunctions are

$$\boxed{X_n(x) = \sin(n\pi x), \quad \lambda_n = n^2\pi^2, \text{ for } n \in \mathbb{N} \setminus \{0\}.}$$

[a] I'm being vague on purpose here since there are lots of technicalities/Analysis involved here. If one must know, the right choice here is $L^2(0,1)$ (ie functions in which $\int_0^1 |f(x)|^2 \mathrm{d}x < +\infty$). Furthermore, for the function to be differentiable, we would require $X \in C^2(0,1)$. This requirement is not strict since we can define this concept called a weak derivative (briefly included in remarks of previous discussion supplements), but this is way outside of the scope of this class. Example as to why this is not enforced: We want to be able to write say a square wave with values 1 for $x \in [1/3, 2/3]$ and 0 otherwise as a linear combination of these eigenfunctions (if these eigenfunctions serve as a basis for this space), but if we imposed the additional requirement on differentiability, then this square wave is not differentiable, and thus is not an element in this space of functions although it does take values 0 at $x = 0$ and $x = 1$.

[b] $\mathbb{N}$ for me includes 0.

Note that the choice of associating the negative sign with $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is crucial in ensuring that it has **no negative**

eigenvalues. (This can be seen from the previous example, where the computed eigenvalues are all positive!) In HW 8 Exercise 2, you will prove that there is a general set of boundary conditions in which the eigenvalues for this differential operator is non-negative!

Next, we shall see that $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is a **self-adjoint** differential operator with respect to its corresponding boundary conditions (which usually is). Before we begin with a formal definition, we first define an inner product:

Definition 59. An inner product $(\cdot,\cdot)$ between two real-valued functions defined on $[a,b]$ is given by[a]

$$(f,g) = \int_a^b f(x)g(x)\,\mathrm{d}x. \tag{363}$$

[a]With the additional condition that these functions f and g are in $L^2(a,b)$, ie $f \in L^2(a,b)$ if $\int_a^b |f(x)|^2 \mathrm{d}x < +\infty$. See footnote from Example 58 on a comment on this. One will have to take the complex conjugate on g if f and g are complex-valued instead.

Here is a formal definition:

Definition 60. A differential operator $\mathcal{L}$ is **self-adjoint** with respect to its *corresponding boundary conditions* if

$$(\mathcal{L}[f], g) = (f, \mathcal{L}[g]) \tag{364}$$

for every $f, g \in C^2$ and satisfying the corresponding boundary conditions.

Remark: Self-adjoint is a property of a differential operator associated with the boundary conditions - that is, if the boundary conditions changes, it might not neccessarily be true that $\mathcal{L}$ remains self-adjoint!

Proposition 61. $\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is self-adjoint with respect to its corresponding boundary conditions on the space of functions defined on $[a,b]$ as long as for each f in that space, we have

- Either $f(a) = 0$ or $f'(a) = 0$, and
- Either $f(b) = 0$ or $f'(b) = 0$. (ie, mixture of Dirichlet and Neumann.) OR,
- We impose periodic boundary conditions $f(a) = f(b), f'(a) = f'(b)$.

In addition, under these boundary conditions, one can also show that the eigenvalues of $\mathcal{L}$ are non-negative. (See HW 8 Exercise 2.)

For a proof in which $a = 0, b = L$ and $f'(0) = f'(L) = 0$, see lecture notes at 8.3.pdf. We shall then include a list of properties that it enjoys:

Proposition 62. A self-adjoint differential operator with respect to its corresponding boundary conditions has the following properties:

(i) Eigenvalues are real.

(ii) Eigenfunctions form a basis for the space of functions in which it is defined for (with the corresponding boundary conditions). Another word for this is that the eigenfunctions are **complete**.

(iii) Distinct eigenvalues corresponds to distinct eigenfunctions.

(iv) There are countably infinitly many eigenvalues.

Though abstract, this will be useful to "shortcut"/justify many computations to come when we are solving a PDE by separation of variables! In view of property (ii) from the above proposition, this means that any function satisfying the boundary conditions can be written as a linear combination of its basis functions (which are the eigenfunctions that we have computed)! In other words, in Example 58, the eigenfunctions are given by $\sin(n\pi x)$ for $n \in \mathbb{N}\backslash\{0\}$. Thus, we can write any function satisfying the boundary conditions (including functions representing initial conditions/boundary conditions) as a linear combination of $\sin(n\pi x)$, ie if $f(0) = f(1) = 0$, we can then write

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \tag{365}$$

for coefficients A_n to be determined. This thus motivates the following concept on Fourier Series.

Fourier Series.

- In Example 58, we have shown that the eigenfunctions for $\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with $X(0) = X(1) = 0$ are given by $X_n(x) = \sin(n\pi x)$. In view of Proposition 62 (ii), these eigenfunctions form a basis for

$$\{ \text{ functions } X \text{ on } [0,1] : X(0) = X(1) = 0\}.$$

  This is why any functions f with $f(0) = f(1) = 0$ can be written as a linear combinations of $\sin(n\pi x)$ for $n \in \mathbb{N} \setminus \{0\}$.

- Similarly, one can show that the eigenfunctions for $\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with $X'(0) = X'(1) = 0$ are given by $X_n(x) = \cos(n\pi x)$ for $n \in \mathbb{N}$, which forms a basis for[25]

$$\{ \text{ functions } X \text{ on } [0,1] : X'(0) = X'(1) = 0\}.$$

- Formally, if we just want to express any function defined on $(0,1)$ (ie excluding the boundary), we could pick $\{\sin(n\pi x) : n \in \mathbb{N} \setminus \{0\}\}$ or $\{\cos(n\pi x) : n \in \mathbb{N}\}$ as the basis functions (ie if we are not concerned with matching boundary values, then either of the basis functions are fine). However, we would not expect the infinite series to converge uniformly on $[0,1]$ since the boundary conditions at $x = 0$ and $x = 1$ are not necessarily satisfied. We call the former, a Fourier sine series, and the latter, a Fourier cosine series.

- In fact, one notes that for the space of function just defined by

$$\{ \text{ functions } X \text{ on } [0,1]\},$$

  an appropriate basis function should be $\{1, \sin(n\pi x), \cos(n\pi x) : n \in \mathbb{N} \setminus \{0\}\}$. We can then write out the full Fourier series, given by

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) + \sum_{n=0}^{\infty} B_n \cos(n\pi x). \tag{366}$$

- Nonetheless, if you want to find solutions to your ODE that satisfies the boundary condition, we should always use the correct basis function obtained by solving for the eigenfunctions X_n with the corresponding boundary conditions!

To match the general form given in lecture, we now consider functions X on $[0,L]$. One can show, following the same argument as in Example 58, that the corresponding eigenfunctions for $X(0) = X(L) = 0$ and $X'(0) = X'(L) = 0$ are

- $X(0) = X(L) = 0$: $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ for $n \geq 1, n \in \mathbb{N}$.
- $X(0) = X(L) = 0$: $X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ for $n \geq 1, n \in \mathbb{N}$, and $X_0(x) = 1$ (with $\lambda_0 = 0$).

These (eigen)functions consisting of 1, $\sin\left(\frac{n\pi x}{L}\right)$ for $n \in \mathbb{N} \setminus \{0\}$, and $\cos\left(\frac{n\pi x}{L}\right)$ for $n \in \mathbb{N} \setminus \{0\}$, satisfy the following orthogonality relations (with respect to the inner product defined above):

Proposition 63. (Orthogonality Relations.)

(i) $\left(\sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right)\right) = \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x = 0$ for all $n \neq m \in \mathbb{N}$.

(ii) $\left(\cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right) = \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) \mathrm{d}x = 0$ for all $n \neq m \neq 0 \in \mathbb{N}$.

(iii) $\left(\cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right) = \int_0^L \cos\left(\frac{n\pi x}{L}\right)^2 \mathrm{d}x = \frac{L}{2}$ for all $n \geq 1$ $(n \in \mathbb{N})$.

(iv) $\left(\sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right) = \int_0^L \sin\left(\frac{n\pi x}{L}\right)^2 \mathrm{d}x = \frac{L}{2}$ for all $n \geq 1$ $(n \in \mathbb{N})$.

(v) $\left(1, \cos\left(\frac{n\pi x}{L}\right)\right) = \int_0^L \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = 0$ for all $n \geq 1$ $(n \in \mathbb{N})$.

(vi) $(1,1) = \int_0^L 1^2 \,\mathrm{d}x = L$. Note that this is L and not $\frac{L}{2}$ as for the other $\sin$ and $\cos$ functions.

[25]This includes the case where $n = 0$, ie $\cos(0) = 1$, the constant function, is also a basis function!

These properties can be proven by direct computation (using Calculus). Some of them are proved in lectures - see 8.1.pdf.

Remark: It makes sense for $\left(1, \cos\left(\frac{n\pi x}{L}\right)\right)$ since they are orthogonal eigenfunctions of the same space. However, it might not be true that $\left(1, \sin\left(\frac{n\pi x}{L}\right)\right)$ (one can see from the Example below).

Suppose we want to write $\phi(x)$ defined on $[0, L]$ in terms of its Fourier sine series. This implies that

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right). \tag{367}$$

To determine the coefficients A_n, multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate from 0 to L. This is equivalent to taking the inner product with respect to $\sin\left(\frac{m\pi x}{L}\right)$ on both sides. We have for $m \geq 1$,

$$\begin{aligned}
\left(\sin\left(\frac{m\pi x}{L}\right), \phi(x)\right) &= \left(\sin\left(\frac{m\pi x}{L}\right), \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\right) \\
\left(\sin\left(\frac{m\pi x}{L}\right), \phi(x)\right) &= \sum_{n=1}^{\infty} A_n \left(\sin\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right) \\
\left(\sin\left(\frac{m\pi x}{L}\right), \phi(x)\right) &= A_m \frac{L}{2} \\
A_m &= \frac{2}{L}\left(\sin\left(\frac{m\pi x}{L}\right), \phi(x)\right) \\
&= \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\phi(x)\,\mathrm{d}x, \quad m \geq 1.
\end{aligned} \tag{368}$$

Here, we have used Proposition 63 (ii) to deduce that all the inner products for different n are 0, unless (by (v)), $m = n$. In that case, the inner product evaluates to $\frac{L}{2}$.[26]
We can do a similar set of derivations for the Fourier cosine series:

$$\phi(x) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right) \tag{369}$$

to obtain

$$\begin{aligned}
B_m &= \frac{2}{L}\int_0^L \cos\left(\frac{m\pi x}{L}\right)\phi(x)\,\mathrm{d}x, \quad m \geq 1, \\
B_0 &= \frac{1}{L}\int_0^L \phi(x)\,\mathrm{d}x.
\end{aligned} \tag{370}$$

Remark: It is important to know the derivation for these formulas as compared to memorizing them, as we might have different eigenfunctions for boundary conditions that are not purely Dirichlet/Neumann.

[26]Pulling the sum outside of the inner product follows from the fact that $(f, g+h) = \int_0^L f(x)(g+h)(x)\,\mathrm{d}x = \int_0^L f(x)g(x)\,\mathrm{d}x + \int_0^L f(x)h(x)\,\mathrm{d}x = (f,g) + (f,h)$, linearity in each of the "slots" of the inner product. If you are worried about justifying interchanging infinite sum with integrals, this might work if we know that the Fourier series of f converges uniformly to f (this depends on "types of convergence" results from Strauss or just Fourier Analysis in general (which might have been covered in 135)).

Example 64. (Strauss 5.1.2i). Let $\phi(x) = x^2$ for $0 \le x \le 1$. Compute its Fourier sine series.

Suggested Solution: We shall use the formulas derived in (368) instead (but one should also understand how these formulas are derived), and put focus on computing some of these integrals that arise.
For $m \ge 1$ and $L = 1$, we have (by applying integration by parts twice),

$$\begin{aligned} A_m &= 2\int_0^1 \sin(m\pi x)x^2 \,\mathrm{d}x \\ &= -\frac{2}{m\pi}\cos(m\pi x)x^2|_0^1 + \int_0^1 \frac{4}{m\pi}\cos(m\pi x)x\,\mathrm{d}x \\ &= -\frac{2}{m\pi}\cos(m\pi) + \frac{4}{m\pi}\int_0^1 \cos(m\pi x)x\,\mathrm{d}x \\ &= -\frac{2}{m\pi}\cos(m\pi) + \frac{4}{m^2\pi^2}\sin(m\pi x)x|_0^1 - \frac{4}{m^2\pi^2}\int_0^1 \sin(m\pi x)\,\mathrm{d}x \\ &= -\frac{2}{m\pi}\cos(m\pi) + \frac{4}{m^2\pi^2}\sin(m\pi) - \frac{4}{m^2\pi^2}\int_0^1 \sin(m\pi x)\,\mathrm{d}x \\ &= -\frac{2}{m\pi}\cos(m\pi) + \frac{4}{m^2\pi^2}\sin(m\pi) + \frac{4}{m^3\pi^3}[\cos(m\pi) - \cos(0)] \\ &= -\frac{2}{m\pi}\cos(m\pi) + \frac{4}{m^2\pi^2}\sin(m\pi) + \frac{4}{m^3\pi^3}[\cos(m\pi) - 1] \end{aligned} \tag{371}$$

The first and third term can be evaluated using the formula

$$\cos(m\pi) = (-1)^m$$

for $m \in \mathbb{N}, m \ge 1$ (check this by substituting $m = 1, 2, 3, \ldots$).

The second term in $\sin(m\pi)$ is zero since

$$\sin(m\pi) = 0$$

for $m \in \mathbb{N}, m \ge 1$.

This implies that

$$A_m = \frac{2(-1)^{m+1}}{m\pi} - \frac{4(1-(-1)^m)}{m^3\pi^3} \tag{372}$$

so

$$x^2 = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n\pi} - \frac{4(1-(-1)^n)}{n^3\pi^3}\right)\sin(n\pi x). \tag{373}$$

If you want to practice, here is another question:

Exercise 65. (Strauss 5.1.2ii.) Let $\phi(x) = x^2$ for $0 \le x \le 1$. Compute its Fourier cosine series.

We are now ready to discuss solving boundary value PDEs by separation of variables.

Solving Boundary Value Problems - Separation of Variables.

We have all the mathematical tools ready, so lets dive right into an example!

Example 66. In lectures, we have solved the wave equation on a bounded domain with Dirichlet boundary conditions. Let us now consider the wave equation with Neumann boundary conditions on a bounded domain $[0, L]$ with $c = 1$ with general initial conditions. Mathematically, this is given by[a]

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) & \text{in } (0,L)\times(0,\infty), \\ u_x(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u_x(L,t) = 0 & \text{on } \{x=L\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } [0,L]\times\{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } [0,L]\times\{t=0\}. \end{cases} \tag{374}$$

Solve this by separation of variables.

Suggested Solution :

Step 1: Look for separable solutions and derive boundary conditions.
Here, we look for **non-trivial** solutions[b] of the form

$$u(x,t) = X(x)T(t)$$

(ie separable). Using the Neumann boundary conditions, this implies that

$$u_x(0,t) = X'(0)T(t) = 0, \quad \text{and } u_x(L,t) = X'(L)T(t) = 0. \tag{375}$$

From the first equation, either $X'(0) = 0$ or $T(t) = 0$ for all $t \geq 0$. However, the latter implies that $u(x,t) = X(x)T(t) = 0$ since T is now the zero function, and we obtain a trivial (zero) “solution” (it might not even satisfy the initial conditions!) to the above PDE. Similarly, one deduces that $X'(L) = 0$. In summary,

$$X'(0) = X'(L) = 0. \tag{376}$$

Plugging this into the PDE, we get

$$\begin{aligned} u_{tt}(x,t) &= X(x)T''(t) \\ u_{xx}(x,t) &= X''(x)T(t) \\ u_{tt}(x,t) - u_{xx}(x,t) &= X(x)T''(t) - X''(x)T(t) = 0. \end{aligned} \tag{377}$$

Dividing both sides of the equation by $X(x)T(t)$,[c] we obtain

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} \tag{378}$$

Since the LHS of (378) only depends on x, and the RHS of (378) only depends on t, then (378) is equals to a constant. One way to understand this is that if x varies while keeping t fixed, it does not change the value on the right. This implies that it must be a constant in x for any given t. Using a similar argument, we then have that it is a constant in t for any given x.[d] Thus, (378) becomes

$$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{T(t)} = \lambda \tag{379}$$

where λ is a constant. In fact, this constant must be a non-negative constant[e], since λ here is viewed as the eigenvalue to the problem $-X''(x) = \lambda X(x)$ with boundary terms $X'(0) = X'(L) = 0$, which we have already shown that the eigenvalues must be non-negative (see HW 8 Exercise 2 or Proposition 61). [f]

Step 2: Solve the corresponding eigenvalue problem in X.

Now, we would like to solve the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x) \\ X'(0) = X'(L) = 0 \end{cases} \tag{380}$$

to obtain the corresponding eigenvalues and more importantly, eigenfunctions. If $\lambda = 0$, then $X(x) = Ax + B$. Using $X'(0) = X'(L) = 0$, we can only determine that $A = 0$. Thus, $X(x) = B$, an arbitrary constant, is an eigenfunction. In particular, the function 1 is an eigenfunction.

For $\lambda > 0$, The general solution is given by

$$\begin{aligned} X(x) &= A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \\ X'(x) &= A\sqrt{\lambda}\cos(\sqrt{\lambda}x) + B\sqrt{\lambda}\sin(\sqrt{\lambda}x). \end{aligned} \tag{381}$$

Using $X'(0) = 0$, this implies that $A\sqrt{\lambda} = 0$. Since $\lambda > 0$, this implies that $A = 0$. With $X'(x) = B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$ left, we use the condition $X'(L) = 0$ to obtain

$$X'(L) = B\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0. \tag{382}$$

Note that it is now possible for this expression to be 0 with $B \neq 0$. This happens when $\sin(\sqrt{\lambda}L) = 0$, or when $\sqrt{\lambda}L = n\pi$ for $n \in \mathbb{N} \setminus \{0\}$, or

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n \in \mathbb{N} \setminus \{0\}. \tag{383}$$

The corresponding eigenfunctions (the functions attached to B in $X(x)$ since $B \neq 0$, and combining with the case when $\lambda > 0$) are

$$X_n(x) = \cos(\sqrt{\lambda_n}L) = \cos\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}, \tag{384}$$

since these functions will satisfy the boundary conditions but are non-zero functions.

Step 3: Solve the corresponding ODE in T.[g]

Going back to (379), this implies that there are only countably finitely many λ (given by λ_n above) that gives a non-zero solution. Thus, for each $n \in \mathbb{N}$, we will be solving the ODE:

$$-T_n''(t) = \lambda_n T_n(t), \tag{385}$$

where we index the function $T(t)$ by n to imply that we are solving a different ODE for different n (due to different values of λ_n). Since $\lambda_n \geq 0$, for $\lambda = 0$ (ie at $n = 0$), we obtain

$$T_0(t) = A_0 t + B_0. \tag{386}$$

Recall that for each n, we are solving a different ODE, so the arbitrary constants are different, and thus are indexed by n.
For $\lambda_n > 0$ (ie for $n \geq 1$), we obtain

$$\begin{aligned} T_n(t) &= A_n\sin(\sqrt{\lambda_n}t) + B_n\cos(\sqrt{\lambda_n}t) \\ &= A_n\sin\left(\frac{n\pi t}{L}\right) + B_n\cos\left(\frac{n\pi t}{L}\right). \end{aligned} \tag{387}$$

Step 4: Obtain general solution by linearity.

By linearity, for each n, the solution $u_n(x,t) = X_n(x)T_n(t)$ is a solution. Thus, a linear combination of these $u_n(x,t)$ is also a solution. This implies that[h]

$$\begin{aligned} u(x,t) &= \sum_{n\in\mathbb{N}} u_n(x,t) \\ &= \sum_{n\in\mathbb{N}} X_n(x)T_n(t) \\ &= (1)(A_0t + B_0) + \sum_{n\in\mathbb{N}, n\geq 1} \cos\left(\frac{n\pi x}{L}\right)\left(A_n\sin\left(\frac{n\pi t}{L}\right) + B_n\cos\left(\frac{n\pi t}{L}\right)\right). \end{aligned} \tag{388}$$

Step 5: Solve for the "Fourier" coefficients.
First, see that (since $\sin(0) = 0, \cos(0) = 1$),

$$\phi(x) = u(x,0) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right). \tag{389}$$

Thus, the B_n are coefficients of the Fourier cosine series. Using (370), we obtain

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\phi(x)\,\mathrm{d}x, \quad n \geq 1, \\ B_0 &= \frac{1}{L}\int_0^L \phi(x)\,\mathrm{d}x. \end{aligned} \tag{390}$$

Next, take $\frac{\partial}{\partial t}$ to obtain

$$\begin{aligned} u_t(x,t) &= A_0 + \sum_{n\in\mathbb{N}, n\geq 1} \cos\left(\frac{n\pi x}{L}\right)\left(\frac{A_n n\pi}{L}\cos\left(\frac{n\pi t}{L}\right) + \frac{-B_n n\pi}{L}\sin\left(\frac{n\pi t}{L}\right)\right) \\ \psi(x) = u_t(x,0) &= A_0 + \sum_{n=1}^{\infty} \frac{A_n n\pi}{L}\cos\left(\frac{n\pi x}{L}\right). \end{aligned} \tag{391}$$

Let $C_0 = A_0$ and $C_n = \frac{A_n n\pi}{L}$, we then obtain the Fourier cosine series again (in C_n). Using (370), we get

$$\begin{aligned} C_n &= \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\psi(x)\,\mathrm{d}x, \quad n \geq 1, \\ C_0 &= \frac{1}{L}\int_0^L \psi(x)\,\mathrm{d}x, \end{aligned} \tag{392}$$

so the A_n's are given by

$$\begin{aligned} A_n &= \frac{2}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\psi(x)\,\mathrm{d}x, \quad n \geq 1, \\ A_0 &= \frac{1}{L}\int_0^L \psi(x)\,\mathrm{d}x. \end{aligned} \tag{393}$$

Thus, we have

$$\boxed{u(x,t) = (1)(A_0 t + B_0) + \sum_{n\in\mathbb{N}, n\geq 1} \cos\left(\frac{n\pi x}{L}\right)\left(A_n \sin\left(\frac{n\pi t}{L}\right) + B_n \cos\left(\frac{n\pi t}{L}\right)\right)} \tag{394}$$

with coefficients A_n, B_n for $n \in \mathbb{N}$ given by (390) and (393).

[a]Technicality: We assume that ϕ and ψ are sufficiently well-behaved so that they can be written in terms of Fourier series. This is usually captured with "assume all functions have sufficient smoothness as required, unless stated otherwise", consistent with remark in one of the footnotes of Example 58.

[b]"Is zero a solution" can be easily checked by substituting it into the PDE such that it satisfies the initial condition. Thus, it makes sense to just search for non-zero solutions.

[c]We did not discuss about the possibility of X and T being 0 at a point. Most books, not even Strauss, discuss this. The most convincing argument I have for you (at least, I am convinced) is that we use the argument in (379) as an intuition, and then come back to (378) and postulate that $X(x)$ are solutions to the eigenvalue problem $X''(x) = -\lambda X(x)$ for λ independent of x and t. Note that philosophically, this makes sense because when we write an ansatz/guess to the PDE, we are already restricting the functions that we are looking for to a smaller space. If it ends up not working, then it implies that either the restriction is too restrictive (say you have an ansatz $u(x,t) = 1$) or there really is no solution. Substitute this into (377), we obtain $X(x)T''(t) = -\lambda T(t)X(x)$. Since this holds for all x and t, we pick a point x^* such that $X(x^*) \neq 0$, and then divide by $X(x^*)$ on both sides to obtain $T''(t) = -\lambda T(t)$. If such a point does not exists, this implies that $X(x) = 0$ for all x, and thus $u(x,t) = X(x)T(t)$, the trivial solution.

[d]If you don't buy this argument, take $\lambda(x,t) = \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}$. Note that $\frac{\partial}{\partial x}\lambda(x,t) = 0$ since $\frac{\partial}{\partial x}\lambda(x,t) = \frac{\partial}{\partial x}\frac{T''(t)}{T(t)} = 0$ is independent of t. Similarly, $\frac{\partial}{\partial t}\lambda(x,t) = \frac{\partial}{\partial t}\frac{X''(t)}{X(t)} = 0$. Thus, $\lambda_x = \lambda_t = 0$ implies that λ is a constant.

[e]You can argue without this heavy machinery as in Example 58, but that requires more computation so this is actually a shortcut!

[f]If the question "can λ ever be complex" ever pops up in your brain, you are thinking like a pure mathematician! Luckily, I have the cure for you - Proposition 62(i) with a similar argument of viewing $-X''(x) = \lambda X(x)$ with $X'(0) = X'(L) = 0$ as an eigenvalue problem.

[g]Note: This is **not** an eigenvalue problem for the simple reason that no boundary conditions were given for $T(t)$. You can't deduce that $T(t)$ is some function in t using the initial condition since they are not necessarily the zero initial condition, so a similar argument in Step 1 does not hold.

[h]We could have written the solution as $C_0(1)(A_0t + B_0) + \sum_{n\in\mathbb{N},n\geq 1} C_n \cos\left(\frac{n\pi x}{L}\right)\left(A_n \sin\left(\frac{n\pi t}{L}\right) + B_n \cos\left(\frac{n\pi t}{L}\right)\right)$ with arbitrary constants C_n in front, but these are absorbed in the A_n's and B_n's, so it doesn't really matter.

Remark: In HW 7 Exercise 5, you are not required to determine the Fourier coefficients. For HW 8, you should always determine the Fourier coefficients if needed.

Remark: Note that separation of variables works really well with Dirichlet/Neumann/Periodic boundary conditions or a mixture of two. For the Dirichlet/Neumann conditions, we usually assume that the value/derivative at the boundary is zero. If this is not zero, we should try to convert the problem to one that solves the zero boundary condition (sounds familiar?). This is done by linearity! The following example illustrates this:

Example 67. Briefly explain how the diffusion equation on $[0,1]$, with constant $D > 0$, can be solved by separation of variables even though the boundary values/derivatives are non-zero.

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } (0,1)\times(0,\infty), \\ u(0,t) = 1 & \text{on } \{x=0\}\times[0,\infty), \\ u_x(1,t) = 2 & \text{on } \{x=1\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } [0,1]\times\{t=0\}. \end{cases} \tag{395}$$

Suggested Solution: By linearity, we would like to solve

$$\begin{cases} (u_1)_t(x,t) = D(u_1)_{xx}(x,t) & \text{in } (0,1)\times(0,\infty), \\ (u_1)(0,t) = 1 & \text{on } \{x=0\}\times[0,\infty), \\ (u_1)_x(1,t) = 2 & \text{on } \{x=1\}\times[0,\infty), \\ (u_1)(x,0) = \psi(x) & \text{on } [0,1]\times\{t=0\} \end{cases} \tag{396}$$

and

$$\begin{cases} (u_2)_t(x,t) = D(u_2)_{xx}(x,t) & \text{in } (0,1)\times(0,\infty), \\ (u_2)(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ (u_2)_x(1,t) = 0 & \text{on } \{x=1\}\times[0,\infty), \\ (u_2)(x,0) = \phi(x)-\psi(x) & \text{on } [0,1]\times\{t=0\} \end{cases} \tag{397}$$

so that $u(x,t) = u_1(x,t) + u_2(x,t)$ solves the original diffusion equation with non-zero boundary conditions. $u_2(x,t)$ can be solved by standard separation of variables technique. This is to hold for any $\psi(x)$.

For u_1, since the boundary values are time-independent, we postulate a steady-state/time-independent solution, i.e, search for solutions by guessing $(u_1)_t = 0$, ie $u_1(x,t)$ is just a function of x. In fact, we can deduce how $u_1(x,t)$ should look like. From $(u_1)_t = D(u_1)_{xx} = 0$, we know that $(u_1)(x,t) = f(x)$. Plug this in to obtain $Df''(x) = 0$ and thus $(u_1)(x,t) = f(x) = Ax + B$.

Using $(u_1)(0,t) = 1$, we get $B = 1$. Using $(u_1)_x(1,t) = 2$, we obtain $A = 2$. This implies that $(u_1)(x,t) = 2x+1$. Now, this function must satisfy the initial condition $(u_1)(x,0) = \psi(x)$, ie $2x+1 = (u_1)(x,0) = 2x+1$. Thus, we can use our freedom of choice for $\psi(x)$ and pick $\psi(x) = 2x+1$ so that $(u_1)(x,0) = \psi(x)$! The solution to the full equation will thus be given by

$$\begin{aligned} u(x,t) &= (u_1)(x,t) + (u_2)(x,t) \\ &= 2x+1+(u_2)(x,t), \end{aligned} \tag{398}$$

where u_2 solves

$$\begin{cases} (u_2)_t(x,t) = D(u_2)_{xx}(x,t) & \text{in } (0,1)\times(0,\infty), \\ (u_2)(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ (u_2)_x(1,t) = 0 & \text{on } \{x=1\}\times[0,\infty), \\ (u_2)(x,0) = \phi(x)-(2x+1) & \text{on } [0,1]\times\{t=0\}. \end{cases} \tag{399}$$

10 Discussion 10.

All functions have sufficient smoothness as required, unless stated otherwise.

Separation of Variables for Inhomogeneous Boundary Conditions.
(This part was present in Supplement 9, but I did not have the time to cover it, and I think I should as it is pretty important.)

Remark: Note that separation of variables works really well with Dirichlet/Neumann/Periodic boundary conditions or a mixture of two. For the Dirichlet/Neumann conditions, we usually assume that the value/derivative at the boundary is zero. If this is not zero, we should try to convert the problem to one that solves the zero boundary condition (sounds familiar?). This is done by linearity! The following example illustrates this:

Example 68. Briefly explain how the diffusion equation on $[0, 1]$, with constant $D > 0$, can be solved by separation of variables even though the boundary values/derivatives are non-zero.

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } (0,1)\times(0,\infty), \\ u(0,t) = 1 & \text{on } \{x=0\}\times[0,\infty), \\ u_x(1,t) = 2 & \text{on } \{x=1\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } [0,1]\times\{t=0\}. \end{cases} \tag{400}$$

Suggested Solution: By linearity, we would like to solve

$$\begin{cases} (u_1)_t(x,t) = D(u_1)_{xx}(x,t) & \text{in } (0,1)\times(0,\infty), \\ (u_1)(0,t) = 1 & \text{on } \{x=0\}\times[0,\infty), \\ (u_1)_x(1,t) = 2 & \text{on } \{x=1\}\times[0,\infty), \\ (u_1)(x,0) = \psi(x) & \text{on } [0,1]\times\{t=0\} \end{cases} \tag{401}$$

and

$$\begin{cases} (u_2)_t(x,t) = D(u_2)_{xx}(x,t) & \text{in } (0,1)\times(0,\infty), \\ (u_2)(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ (u_2)_x(1,t) = 0 & \text{on } \{x=1\}\times[0,\infty), \\ (u_2)(x,0) = \phi(x)-\psi(x) & \text{on } [0,1]\times\{t=0\} \end{cases} \tag{402}$$

so that $u(x,t) = u_1(x,t) + u_2(x,t)$ solves the original diffusion equation with non-zero boundary conditions. $u_2(x,t)$ can be solved by standard separation of variables technique. This is to hold for any $\psi(x)$, and we will use this freedom of choice for ψ so that any $u_1(x,t)$ that we've found can satisfy the initial condition by picking the correct $\psi(x)$.

For u_1, since the boundary values are time-independent, we postulate a steady-state/time-independent solution, i.e, search for solutions by guessing $(u_1)_t = 0$, ie $u_1(x,t)$ is just a function of x. In fact, we can deduce how $u_1(x,t)$ should look like. From $(u_1)_t = D(u_1)_{xx} = 0$, we know that $(u_1)(x,t) = f(x)$. Plug this in to obtain $Df''(x) = 0$ and thus $(u_1)(x,t) = f(x) = Ax + B$.

Using $(u_1)(0,t) = 1$, we get $B = 1$. Using $(u_1)_x(1,t) = 2$, we obtain $A = 2$. This implies that $(u_1)(x,t) = 2x+1$. Now, this function must satisfy the initial condition $(u_1)(x,0) = \psi(x)$, ie $2x+1 = (u_1)(x,0) = 2x+1$. Thus, we can use our freedom of choice for $\psi(x)$ and pick $\psi(x) = 2x+1$ so that $(u_1)(x,0) = \psi(x)$! The solution to the full equation will thus be given by

$$\begin{aligned} u(x,t) &= (u_1)(x,t) + (u_2)(x,t) \\ &= 2x+1+(u_2)(x,t), \end{aligned} \tag{403}$$

where u_2 solves

$$\begin{cases} (u_2)_t(x,t) = D(u_2)_{xx}(x,t) & \text{in } (0,1)\times(0,\infty), \\ (u_2)(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ (u_2)_x(1,t) = 0 & \text{on } \{x=1\}\times[0,\infty), \\ (u_2)(x,0) = \phi(x)-(2x+1) & \text{on } [0,1]\times\{t=0\}. \end{cases} \tag{404}$$

Laplace's and Poisson's Equation.
Roughly speaking, Laplace's equation is given by

$$-\nabla^2 u = 0$$

where $\nabla^2 = \nabla \cdot \nabla$ (divergence of gradient) is called the Laplacian operator. The Poisson's equation includes a forcing term $f(x)$, and thus is given by

$$-\nabla^2 u = f.$$

These equations are derived by searching for time-independent solutions to the diffusion equation (ie $u_t = Du_{xx}$ and set $u_t = 0$ since we are searching for a solution $u(x,t) \equiv u(x)$ only, so $u_t = 0$). For this class, we will deal with the 2 and 3 dimensional case only. (The one-dimensional case is just an ODE!)

On curvilinear domains in which the unknown function only depends on one of the curvilinear[27] coordinates, one can rewrite the Laplacian in the corresponding coordinate system and solve an ODE. Thus, although it is instructive to know how to write the Laplacian in 2 and 3 dimensions in terms of the spherical and cylindrical coordinates, you are not required to remember how to for this class. We shall look at an example below, in which a similar idea follows from HW 9 Exercise 4.

Example 69. (Strauss 6.1.4, modified.) Recall that in spherical polar coordinates (r, θ, ϕ), the Laplacian of a function $u(r)$ is given by

$$\nabla^2 u(r) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}}{\mathrm{d}r} u(r)\right). \tag{405}$$

Solve the Laplace's equation $-\nabla^2 u = 0$ in the spherical shell $1 < r < 2$ where $u(1) = 1$ and $u(2) = 2$.

Suggested Solution: By searching for a spherical symmetric solution $u(x,y,z) = u(r)$. we then have

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}}{\mathrm{d}r} u(r)\right) = 0. \tag{406}$$

Multiplying by r^2 and integrating with respect to r, we have

$$r^2 \frac{\mathrm{d}}{\mathrm{d}r} u(r) = C' \tag{407}$$

for an arbitrary constant C'. Dividing both sides by r^2 (you can because we are solving u for $1 < r < 2$ so $r > 0$), we have

$$u'(r) = \frac{C'}{r^2}. \tag{408}$$

Integrating with respect to r again, we obtain

$$u(r) = \frac{C}{r} + D \tag{409}$$

where C and D are arbitrary constants (in fact, $C = -C'$). Using the fact that $u(1) = 1$ and $u(2) = 2$, we obtain

$$\begin{cases} C + D = 1 \\ \frac{1}{2}C + D = 2. \end{cases} \tag{410}$$

Solving them, we have $C = -2$ and $D = 3$. The solution to Laplace's equation that satisfies the boundary condition on a spherical shell $1 < r < 2$ is given by

$$u(r) = -\frac{2}{r} + 3.$$

[27] A general name for spherical, cylindrical, elliptical, etc.

For Laplace's and Poisson's equation on a rectangular/cuboid domain, we shall begin our discussion with a few observations for the 2D case (a similar logic holds in 3D):

- Most of the time, we are given boundary conditions at $x = a, b$ and $y = c, d$ (considering that we have a rectangle $[a, b] \times [c, d]$). Each of these will corresponds to the boundary conditions for the separated variables $X(x)$ and $Y(y)$.

- For separation of variables for Laplace's equation, it will always work. For instance, if we are solving $-\nabla^2 u = 0$, look for separation of variables $u(x, y) = X(x)Y(y)$. We then obtain $-X''(x)Y(y) - X(x)Y''(y) = 1$. If we follow a similar strategy of dividing by $X(x)Y(y)$, we have

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda.$$

  The correct strategy to solve the two ODEs depends on the boundary conditions.

- As long as we have Dirichlet/Neumann/mixed/Periodic boundary conditions for **either** of the variables, we can use the general strategy for separation of variables for that variable. i.e Suppose without loss of generality that we have these boundary conditions for $X(x)$. Solve for eigenfunctions and eigenvalues, substitute the eigenvalues back to the separated ODE for $Y(y)$. Now, obtain the full solution, substitute the boundary conditions at $y = c$ and $y = d$ to obtain two separate Fourier series in which the coefficients can be solved.

- If the above point does not hold (ie we have fully/somewhat fully inhomogeneous boundary conditions), remember that we can always split the Laplace's equation into two by linearity, so that we are solving two separate instances of the equation, one with boundary conditions in $X(x)$ and the zero condition for $Y(y)$; and the other with boundary conditions in $Y(y)$ and the zero condition for $X(x)$. We then repeat the previous pointer twice, and add the two solutions up by linearity.

- Now, we have a general strategy for Laplace's equation! For Poisson's equation, note that the equation does not "separate". For instance, if we are solving $-\nabla^2 u = 1$, look for separation of variables $u(x, y) = X(x)Y(y)$. We then obtain $-X''(x)Y(y) - X(x)Y''(y) = 1$. If we follow a similar strategy of dividing by $X(x)Y(y)$, we have

$$-\frac{X''(x)}{X(x)} - \frac{Y''(y)}{Y(y)} = \frac{1}{X(x)Y(y)}.$$

  The equation above is not separated into "separate variables" on LHS and RHS of the equation. Thus, strictly speaking, separation of variables does not work for Poisson's equation.

- The strategy will be similar to that in Example 68, by considering a "translation" so that the forcing term f is now zero, but we transfer this forcing term f to the boundary conditions. This will be elaborated and guided in HW 8 Exercise 5. The following example also covers this (with a different forcing function and a slight variant of the strategy covered in HW 8 Exercise 5).

Example 70. Fully inhomogeneous Poisson's equation on a square.[a] Solve the following Poisson's equation on a square:

$$\begin{cases} -u_{xx}(x,y) - u_{yy}(x,y) = x & \text{in } (0,1)\times(0,1), \\ u(0,y) = 1 & \text{on } \{x=0\}\times[0,1], \\ u(1,y) = 1 & \text{on } \{x=1\}\times[0,1], \\ u(x,0) = 1 & \text{on } [0,1]\times\{y=0\}, \\ u(x,1) = 1 & \text{on } [0,1]\times\{y=1\}. \end{cases} \tag{411}$$

Suggested Solution:

Step 1: Transferring Forcing Term to Boundary Conditions. Following the remarks from the previous page, one should try to "transfer" the forcing term $f(x,y) = x$ to the boundary conditions. This can be done by observing that

$$\begin{aligned} -u_{xx} - u_{yy} &= x \\ -(u_{xx} + x) - u_{yy} &= 0 \\ -\left(u + \frac{x^3}{6} + Ax + B\right)_{xx} - u_{yy} &= 0 \\ -\left(u + \frac{x^3}{6} + Ax + B\right)_{xx} - \left(u + \frac{x^3}{6} + Ax + B\right)_{yy} &= 0. \end{aligned} \tag{412}$$

Here, we obtain $\frac{x^3}{6} + Ax + B$ by integrating x with respect to x twice (A and B are the resulting arbitrary constants). Furthermore, this does not depends on y, so any partial derivatives in y will cause the term to vanish. Then, we use the substitution $w(x,y) = u(x,y) + \frac{x^3}{6} + Ax + B$. Thus, we have

- $w(x,0) = u(x,0) + \frac{x^3}{6} + Ax + B = \frac{x^3}{6} + Ax + B + 1.$
- $w(x,1) = u(x,1) + \frac{x^3}{6} + Ax + B = \frac{x^3}{6} + Ax + B + 1.$
- $w(0,y) = u(0,y) + \frac{0^3}{6} + A(0) + B = B + 1.$
- $w(1,y) = u(1,y) + \frac{1^3}{6} + A(1) + B = A + B + \frac{7}{6}.$

Note that since A and B are arbitrary (ie we could have picked any A and B to do the substitution and it will still work), I'll pick $B = -1$ (so that $B + 1 = 0$ to simplify a lot of these terms) and $A = 0$ (so that we have lesser terms to deal with). This implies that we are now solving:

$$\begin{cases} -w_{xx}(x,y) - w_{yy}(x,y) = 0 & \text{in } (0,1)\times(0,1), \\ w(0,y) = 0 & \text{on } \{x=0\}\times[0,1], \\ w(1,y) = \dfrac{1}{6} & \text{on } \{x=1\}\times[0,1], \\ w(x,0) = \dfrac{x^3}{6} & \text{on } [0,1]\times\{y=0\}, \\ w(x,1) = \dfrac{x^3}{6} & \text{on } [0,1]\times\{y=1\}. \end{cases} \tag{413}$$

Remark: In HW 8 Exercise 5, if you follow through the steps there, you will realize that this is roughly the same as those given.

Step 2: Separating Laplace's Equation into two, with homogeneous in one direction for each.

Let $w(x,y) = w_1(x,y) + w_2(x,y)$, with

$$\begin{cases} -(w_1)_{xx}(x,y) - (w_1)_{yy}(x,y) = 0 & \text{in } (0,1)\times(0,1), \\ (w_1)(0,y) = 0 & \text{on } \{x=0\}\times[0,1], \\ (w_1)(1,y) = 0 & \text{on } \{x=1\}\times[0,1], \\ (w_1)(x,0) = \dfrac{x^3}{6} & \text{on } [0,1]\times\{y=0\}, \\ (w_1)(x,1) = \dfrac{x^3}{6} & \text{on } [0,1]\times\{y=1\}, \end{cases} \tag{414}$$

and

$$\begin{cases} -(w_2)_{xx}(x,y) - (w_2)_{yy}(x,y) = 0 & \text{in } (0,1)\times(0,1), \\ (w_2)(0,y) = 0 & \text{on } \{x=0\}\times[0,1], \\ (w_2)(1,y) = \dfrac{1}{6} & \text{on } \{x=1\}\times[0,1], \\ (w_2)(x,0) = 0 & \text{on } [0,1]\times\{y=0\}, \\ (w_2)(x,1) = 0 & \text{on } [0,1]\times\{y=1\}. \end{cases} \tag{415}$$

Step 3: Solve the resulting two separate Laplace's equations.

For (415), by separation of variables, let $(w_1)(x,y) = X(x)Y(y)$ for non-trivial w_1. Substituting this to the Laplace's equation and dividing by $X(x)Y(y)$, we have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda.$$

Similar to Discussion Supplement 9, $(w_2)(x,0) = 0$ means $X(x)Y(0) = 0$ for all $y \in [0,1]$. For non-trivial w_2, X can't be the zero function so we must have $Y(0) = 0$. Similarly, we have $Y(1) = 0$. By viewing $-\frac{Y''(y)}{Y(y)} = \lambda$, we must have $\lambda \geq 0$ since $-\frac{d^2}{dx^2}$ is a self-adjoint differential operator with non-negative eigenvalues.

If $\lambda = 0$, then $Y''(y) = 0$ so $Y(y) = Ay + B$. Using $Y(0) = Y(1) = 0$, one can deduce that $A = B = 0$ and thus $Y(y) \equiv 0$. We thus obtain the trivial solution for w_1, which is not what we want. Thus, we have for $\lambda > 0$,

$$Y''(y) = -\lambda Y(y), \quad Y(0) = Y(1) = 0.$$

The solution is given by $Y(y) = A\sin(\sqrt{\lambda}y) + B\cos(\sqrt{\lambda}y)$. Substituting $Y(0) = 0$ gives $B = 0$. Substituting $Y(1) = 0$ gives $Y(1) = A\sin(\sqrt{\lambda}) = 0$, in which the solution is given by $\sqrt{\lambda} = n\pi$, $\lambda_n = n^2\pi^2$ for $n \geq 1, n \in \mathbb{N}$. The corresponding eigenfunctions are $Y_n(y) = \sin(n\pi y)$.

For each λ_n, we solve a corresponding ODE in X. This is given by

$$X_n''(x) = \lambda_n X_n(x) = n^2\pi^2 X_n(x).$$

The solutions are given by

$$X_n(x) = C_n' \sinh(n\pi x) + D_n' \cosh(n\pi x).$$

One minor disadvantage of this form is that though $\sinh$ vanishes at 0, $\cosh$ never vanishes. One can then show that (this is also shown in lectures, 9.3.pdf) that an equivalent form is given by (with the 1 easily replaceable by any constants, with possibly different arbitrary constants),

$$X_n(x) = C_n \sinh(n\pi x) + D_n \sinh(n\pi(1-x)).$$

The full solution is thus given by

$$\begin{aligned} (w_1)(x,y) &= \sum_{n\geq 1} X_n(x)Y_n(y) \\ &= \sum_{n\geq 1} \sin(n\pi y)(C_n \sinh(n\pi x) + D_n \sinh(n\pi(1-x)). \end{aligned} \tag{416}$$

To determine the Fourier coefficients C_n and D_n, we appeal to the inhomogeneous boundary conditions in x (ie $(w_2)(0,y) = 0$ and $(w_2)(0,y) = \frac{1}{6}$). Using $(w_2)(0,y) = 0$, we have

$$0 = (w_2)(0,y) = \sum_{n=1}^{\infty} D_n \sinh(n\pi)\sin(n\pi y). \tag{417}$$

Let $E_n = D_n \sinh(n\pi)$, we have

$$\sum_{n=1}^{\infty} E_n \sin(n\pi y) = 0. \tag{418}$$

We can thus determine the coefficients E_n by a standard Fourier series argument. For this case, we have $E_m = \frac{(0,\sin(m\pi y))}{(\sin(m\pi y),\sin(m\pi y))} = 0$ for every $m \geq 1, m \in \mathbb{N}$. (See Discussion Supplement 9.) This implies that $D_n = 0$ too for each $n \geq 1$.

Using $(w_2)(1,y) = \frac{1}{6}$, we obtain

$$\frac{1}{6} = (w_2)(1,y) = \sum_{n=1}^{\infty} C_n \sinh(n\pi)\sin(n\pi y). \tag{419}$$

Let $F_n = C_n \sinh(n\pi)$, we have

$$\sum_{n=1}^{\infty} F_n \sin(n\pi y) = \frac{1}{6}. \tag{420}$$

Then, we have $F_n = \frac{(1/6,\sin(n\pi y))}{(\sin(n\pi y),\sin(n\pi y))}$ for $n \geq 1$, $n \in \mathbb{N}$. One can compute

$$(\sin(n\pi y),\sin(n\pi y)) = \int_0^1 \sin^2(n\pi y)\mathrm{d}y = \frac{1}{2},$$

and

$$(1/6,\sin(n\pi y)) = \frac{1}{6}\int_0^1 \sin(n\pi y)\mathrm{d}y = -\frac{1}{6n\pi}\cos(n\pi y)|_{y=0}^{y=1} = \frac{1}{6n\pi}(1-(-1)^n).$$

This implies that $F_n = \frac{1-(-1)^n}{3n\pi}$. This thus implies that $C_n = F_n/(\sinh(n\pi)) = \frac{1-(-1)^n}{3n\pi\sinh(n\pi)}$ for $n \geq 1$. Combining all that we have, the full solution is given by

$$(w_1)(x,y) = \sum_{n=1}^{\infty} \frac{1-(-1)^n}{3n\pi\sinh(n\pi)}\sin(n\pi y)\sinh(n\pi x). \tag{421}$$

We shall skip solving w_1 as this follows from an exact same procedure! (You can treat it as an exercise to see if you have understood the idea of separation of variables.)

Step 4: Combine them all. Recall that $w(x,y) = (w_1)(x,y) + (w_2)(x,y)$, and this was obtained after a substitution $u(x,y) = w(x,y) - \frac{x^3}{6} + 1$ (Recall $A = 0$ and $B = -1$). Thus, we have

$$u(x,y) = 1 - \frac{x^3}{6} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{3n\pi\sinh(n\pi)}\sin(n\pi x)\sinh(n\pi y) + w_1(x,y), \tag{422}$$

where you have to put in w_1 obtained by solving its corresponding Laplace's equation.

[a] Marcus's remark in lecture is that something as tedious as this will not appear in the exam, though personally, I think it makes more sense to cover a tedious example in Discussion to see how different cases can be handled, and as a practice for separation of variables. Thus, if any subset of these cases were given, you would know how to solve it.

Remark: The solution form (422) is consistent with the second last pointer before the Example - separation of variables will not work. Literally, one can see that the solution cannot be written as the product of two functions of their individual variables! (We can do that for w_1 and w_2, but not when combined together, and definitely not when we have added (not multiply) a term in only the x variable.)

11 Finals Revision.

Practice Questions - Set 1

As per usual, we assume that all unknown functions are sufficiently smooth, unless stated otherwise.

1. (10 points) Consider the PDE consisting of the unknown function $u(x, y)$ given by

$$u_x + 4xu_y = 0, \tag{423}$$

with boundary conditions

$$\begin{cases} u(x,0) = e^{-x} & \text{for all } x \geq 0 \\ u(0,y) = y^4 & \text{for all } y \geq 0. \end{cases} \tag{424}$$

Solve for $u(x, y)$ for $x, y > 0$.

2. Consider the following PDE:

$$u_{xx}(x,t) - 3u_{xt}(x,t) - 4u_{tt}(x,t) = 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty). \tag{425}$$

(a) (4 points) Classify the type of second-order PDE in (425) (ie determine if it is elliptic, parabolic, or hyperbolic at each point $(x, t) \in \mathbb{R} \times (0, \infty)$).

(b) (8 points) Find the general solution $u(x, t)$ to (425).

(c) (5 points) Hence or otherwise, find the general solution $u(x, t)$ to the following PDE

$$u_{xx}(x,t) - 3u_{xt}(x,t) - 4u_{tt}(x,t) = x - t \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty). \tag{426}$$

3. Consider the diffusion equation on a bounded domain with arbitrary initial data $\phi(x)$ on $x \in [0, L]$ for some finite $L > 0$, and arbitrary boundary data $f(t)$ and $g(t)$ on $x = 0$ and $x = L$ respectively, given by

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } \mathbb{R} \times (0,L), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t=0\}, \\ u(0,t) = f(t) & \text{on } \{x=0\} \times (0,\infty), \\ u(b,t) = g(t) & \text{on } \{x=b\} \times (0,\infty). \end{cases} \tag{427}$$

(a) (3 points) We know that the PDE above satisfies the maximum principle. Write down sufficient condition(s) on $\phi(x)$, $f(t)$, and $g(t)$ such that the solution $u(x, t)$ is bounded for all $(x, t) \in \mathbb{R} \times (0, \infty)$.

(b) (7 points) Use the maximum principle to prove that the solution to (427) is unique if it exists.

4. (15 points) The *Stokes' First Problem* is one of a handful of known exact solutions of the Navier-Stokes equation, which describes how fluid lying on a flat plate starts to move when the plate is set in motion. Let $u(z,t)$ denote the velocity of fluid in the $x-$direction located at a perpendicular distance z from the plate. The PDE satisfied by $u(z,t)$ is given by

$$\begin{cases} u_t(z,t) = \nu u_{zz}(z,t) & \text{in } (0,\infty)\times(0,\infty), \\ u(z,0) = 0 & \text{on } [0,\infty)\times\{t=0\}, \\ u(0,t) = U & \text{on } \{z=0\}\times(0,\infty), \\ u(+\infty,t) = 0 & \text{on } \{z=+\infty\}\times(0,\infty). \end{cases} \tag{428}$$

Here, U represents the velocity of the plate and ν represents the kinematic viscosity of the fluid, which are both positive constants. Find the functional form of the similarity solution to (428) of the form

$$u(z,t) = H(t)f\left(\frac{z}{W(t)}\right).$$

(a) In your solution, you should determine the functions H and W.

(b) You should also determine the ODE satisfied by f, together with the corresponding boundary conditions for f.

5.

(a) (9 points) Solve the following wave equation for $u(x,t)$ on the half-line with Dirichlet boundary conditions, given by

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ u(x,0) = \sin(x) & \text{on } [0,\infty)\times\{t=0\}, \\ u_t(x,0) = x & \text{on } [0,\infty)\times\{t=0\}, \\ u(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty). \end{cases} \tag{429}$$

You will need to simplify your answer as far as possible.

(b) (6 points) Hence or otherwise, solve the following wave equation for $v(x,t)$ on the half-line with a source term at $x=0$, given by

$$\begin{cases} v_{tt}(x,t) = v_{xx}(x,t) & \text{in } (0,\infty)\times(0,\infty), \\ v(x,0) = \sin(x) & \text{on } [0,\infty)\times\{t=0\}, \\ v_t(x,0) = x & \text{on } [0,\infty)\times\{t=0\}, \\ v(0,t) = t^3 & \text{on } \{x=0\}\times[0,\infty). \end{cases} \tag{430}$$

For this part of the question, writing down the correct integral(s) to be evaluated will suffice.

Hint: Consider an appropriate substitution.

6. (20 points) Solve the following Laplace's equation on a square $[0,1]\times[0,1]$ with inhomogeneous boundary conditions:

$$\begin{cases} -u_{xx}(x,y) - u_{yy}(x,y) = 0 & \text{in } (0,1)\times(0,1), \\ u(0,y) = 0 & \text{on } \{x=0\}\times[0,1], \\ u(1,y) = 1 & \text{on } \{x=1\}\times[0,1], \\ u(x,0) = x & \text{on } [0,1]\times\{y=0\}, \\ u(x,1) = 0 & \text{on } [0,1]\times\{y=1\}. \end{cases} \tag{431}$$

7. (15 points) Consider the diffusion equation with arbitrary initial data $\phi(x)$:

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{432}$$

By considering the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} u^2(x,t)\mathrm{d}x, \tag{433}$$

prove that the above diffusion equation has at most one $u \in C^2$ solution, with u vanishing as $x \to +\infty$, and u_x vanishing as $x \to -\infty$.

Practice Questions - Set 2

1. Let us consider an observer in $2D$, with the observer's position represented in Cartesian coordinates (x, y), and the observer's path parametrized by time t (that is, $\mathbf{r}(t) = (x(t), y(t))$). With respect to a stationary observer, the density of matter $u(x, y, t)$ is described by the PDE:

$$u_t + u_x + e^{-t}u_y = 0 \tag{434}$$

for $t > 0$, and $x, y \in \mathbb{R}$. Assume that we have an initial distribution of matter at $t = 0$, given by $u(x, y, 0) = e^{-x^2-y^2}$ for $(x, y) \in \mathbb{R}^2$.

(i) (3 points) Given a position $(x(t), y(t))$ of the observer at time t, write down the velocity vector $\mathbf{r}'(t)$ describing the velocity of the observer such that the density of matter does not change relative to this observer.

(ii) (7 points) Hence, solve the PDE (434) for $u(x, y, t)$, with $(x, y, t) \in \mathbb{R}^2 \times (0, \infty)$.

2. (15 points) Consider the *telegrapher's equation* with arbitrary initial data $\phi(x)$ and $\psi(x)$, given by

$$\begin{cases} u_{tt} + 2u_t = 3u_{xx} & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = \phi(x) & \text{on } \mathbb{R} \times \{t = 0\}, \\ u_t(x, 0) = \psi(x) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{435}$$

In addition, consider the energy function

$$E(t) = \frac{1}{2} \int_0^\infty (u_t)^2(x, t) + \alpha(u_x)^2(x, t)\mathrm{d}x. \tag{436}$$

With the help of the energy function and an appropriate value of α, prove that the solution to the telegrapher's equation with u_t and u_x vanishing as $x \to \pm\infty$ in (435) is unique if it exists.

3. (15 points) Solve the following diffusion equation for $u(x, t)$ on the half-line with Neumann boundary conditions, given by

$$\begin{cases} u_t(x, t) = u_{xx}(x, t) + 2022 & \text{in } (0, \infty) \times (0, \infty), \\ u(x, 0) = e^{-x^2} & \text{on } [0, \infty) \times \{t = 0\}, \\ u_x(0, t) = 0 & \text{on } \{x = 0\} \times (0, \infty). \end{cases} \tag{437}$$

You will need to simplify your answer as far as possible.

4. A droplet spreads under gravity on a horizontal surface. The height of the droplet is given by a function $h(x,t)$. It can be shown that the height of the droplet $h(x,t)$ evolves according to the following simplified version of the *porous medium equation*:

$$\frac{\partial h}{\partial t} = \frac{1}{3}\frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) \quad \text{in } (x,t) \in (-L,L) \times (0,\infty) \tag{438}$$

for some finite $L > 0$ independent of time. In addition, we shall also impose the following physical boundary conditions: $h(\pm L, t) = 0$.

(a) (7 points) Prove that the quantity $V(t) = \int_{-L}^{L} h(x,t)\mathrm{d}x$ representing the total volume of the droplet is a constant function in t.

(b) (8 points) Find the functional form of the similarity solution to (438) of the form

$$h(x,t) = H(t) f\left(\frac{x}{W(t)}\right),$$

where H and W are functions to be determined. For this problem, you do not have to determine the ODE satisfied by f. You do not need to determine the corresponding boundary conditions too.

5. (20 points) Solve the following PDE on a bounded domain $x \in (0,1)$ with $t > 0$ for $u(x,t)$ by separation of variables:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - u(x,t) & \text{in } (0,1) \times (0,\infty), \\ u(x,0) = x & \text{on } [0,1] \times \{t = 0\}, \\ u(0,t) = 0 & \text{on } \{x = 0\} \times [0,\infty), \\ u(1,t) = 0 & \text{on } \{x = 1\} \times [0,\infty). \end{cases} \tag{439}$$

Hint: 1 is a constant independent of both x and t.

6. (15 points) Solve the following Poisson's equation on a square $[0,1] \times [0,1]$ with homogeneous boundary conditions for $u(x,y)$:

$$\begin{cases} -u_{xx}(x,y) - u_{yy}(x,y) = x^2 & \text{in } (0,1) \times (0,1), \\ u(0,y) = 0 & \text{on } \{x = 0\} \times [0,1], \\ u(1,y) = 0 & \text{on } \{x = 1\} \times [0,1], \\ u(x,0) = 0 & \text{on } [0,1] \times \{y = 0\}, \\ u(x,1) = 0 & \text{on } [0,1] \times \{y = 1\}. \end{cases} \tag{440}$$

For this problem, you do not have to compute the Fourier coefficients explicitly, though you should write down the formula and the corresponding integral that one should do to obtain the relevant coefficients.

Hint: Look for solutions that do not depend on y, ie $u(x,y) = f(x)$, that solves the Poisson's equation. Then, determine f and use the substitution $u(x,y) = f(x) + w(x,y)$.

7. (10 points) Consider the diffusion equation on a bounded domain, solved with different initial data $\phi_i(x)$ on $x \in [0, L]$ with some finite $L > 0$ for $i \in \{1, 2\}$, with the same boundary data $f(t)$ and $g(t)$ on $x = 0$ and $x = L$ respectively, given by

$$\begin{cases} (u_i)_t(x,t) = Du_{xx}(x,t) & \text{in } \mathbb{R} \times (0, L), \\ (u_i)(x,0) = \phi_i(x) & \text{on } \mathbb{R} \times \{t = 0\}, \\ (u_i)(0,t) = f(t) & \text{on } \{x = 0\} \times (0, \infty), \\ (u_i)(b,t) = g(t) & \text{on } \{x = b\} \times (0, \infty). \end{cases} \tag{441}$$

Prove that the solutions to (441) are stable with respect to the initial data in the following sense:

$$\max_{x \in [0,L]} |u_1(x,t) - u_2(x,t)| \le \max_{x \in [0,L]} |\phi_1(x) - \phi_2(x)| \tag{442}$$

for any $t > 0$.

Hint: Apply Maximum principle.

More problems:

1. (15 points) Consider the following modified wave equation with $A \geq 0$:

$$\begin{cases} (u_{tt} + Au_t - u_{xx} + u)(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{443}$$

By considering the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2(x,t) + (u_x)^2(x,t) + u^2(x,t)\mathrm{d}x, \tag{444}$$

prove that the above modified wave equation has at most one $u \in C^2$ solution, with u_x and u_t vanishing as $x \to \pm\infty$.

References

[1] Walter A Strauss. Partial differential equations: An introduction. John Wiley & Sons, 2007.